Balancing legged robots

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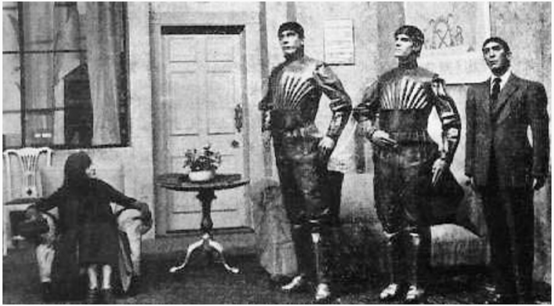
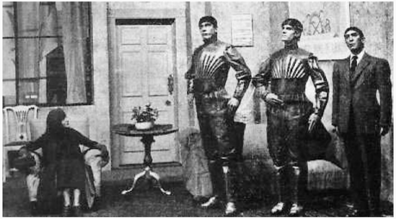
Foreword



Rossumovi Univerzální Roboti, Karel Čapek, 1921.

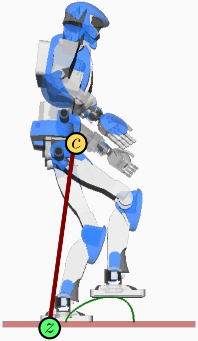
DARPA Robotics Challenge, Finals, 2015.



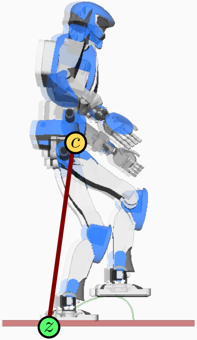
[1] Emanuel Todorov. “Goal directed dynamics”. In: *2018 IEEE International Conference on Robotics and Automation*. IEEE. 2018, pp. 2994–3000.

Physics of balancing

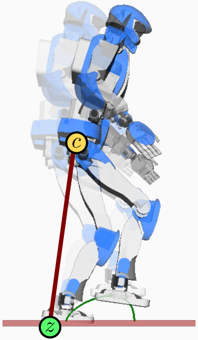
Motivation



Plan$(t + \Delta t)$



Simulation$(t + \Delta t)$

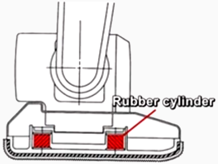


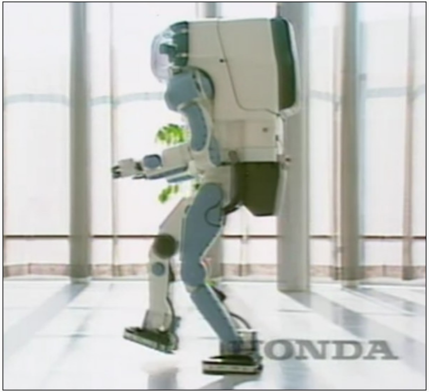
Real$(t + \Delta t)$

Honda P2

Public demonstration in 1998:

- “Zero” Moment Point (ZMP) control
- Ground reaction force control
- Impact absorption:





[2] Kazuo Hirai, Masato Hirose, Yuji Haikawa, and Toru Takenaka. “The development of Honda humanoid robot”. In: *IEEE International Conference on Robotics and Automation*. 1998.

Physical model that makes it walk

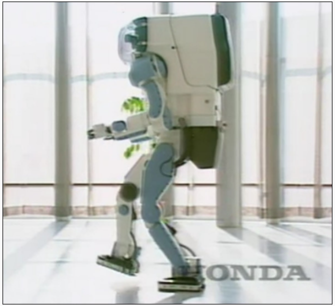


Figure 1: Honda P2 walking

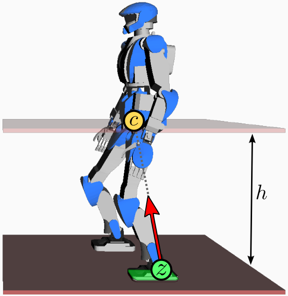


Figure 2: Model it relies on

Centroidal dynamics

Whole-body dynamics:

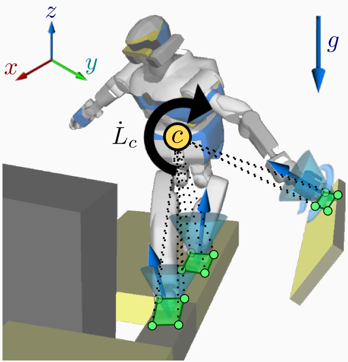
$$M\ddot{q} + N = S^T \tau + J^T f$$

Actuated joints / floating base:

$$M_a \ddot{q} + N_a = \tau + J_a^T f$$
$$M_u \ddot{q} + N_u = J_u^T f$$

Centroidal dynamics:

$$\ddot{c} = g + \frac{1}{m} \sum_{i \in \text{contacts}} f_i$$
$$\dot{L}_c = \sum_{i \in \text{contacts}} (p_i - c) \times f_i$$



[3] Hervé Audren, Joris Vaillant, Abderrahmane Kheddar, Adrien Escande, Kenji Kaneko, and Eiichi Yoshida. "Model preview control in multi-contact motion-application to a humanoid robot". In: *IEEE/RSJ International Conference on Intelligent Robots and Systems*. 2014.

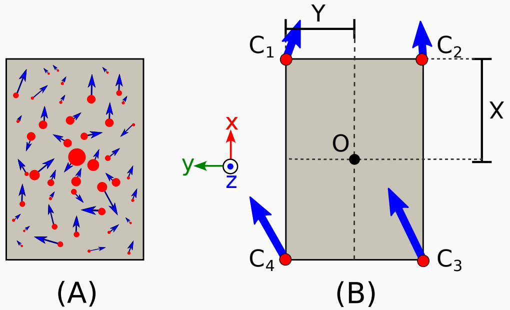


Figure 3: (A) Continuous shearing/pressure distribution under a contact surface. (B) Equivalent discrete force distribution, *i.e.* producing the same wrench.

[4] Stéphane Caron, Quang-Cuong Pham, and Yoshihiko Nakamura. “Stability of Surface Contacts for Humanoid Robots: Closed-Form Formulae of the Contact Wrench Cone for Rectangular Support Areas”. In: *IEEE International Conference on Robotics and Automation*. 2015.

The **center of pressure (CoP)** is the point p on the contact surface where the resultant of *pressure* forces is applied:

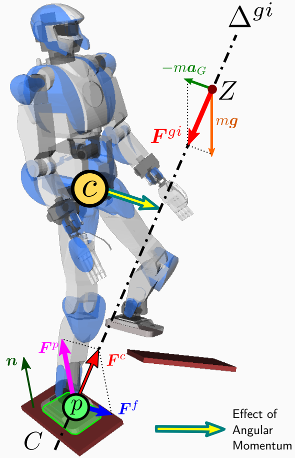
$$p = \int_{(x,y)\in\mathcal{S}} \begin{bmatrix} x \\ y \end{bmatrix} f_z(x,y)\mathrm{d}x\mathrm{d}y$$

Zero-tilting Moment Point (ZMP)

The ZMP is a point where the moment of the contact wrench is aligned with the contact normal n.

The ZMP axis intersects the contact surface at the CoP.

Informally, the net contact force is applied at the ZMP.



[5] P. Sardain and G. Bessonnet. "Forces acting on a biped robot. center of pressure-zero moment point". In: *IEEE Transactions on Systems, Man and Cybernetics, Part A: Systems and Humans* 34.5 (2004).

Reduction to a point-mass model

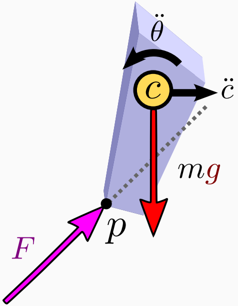


Figure 4: Net contact force does not go through CoM c
$\Rightarrow \dot{L}_c = I\ddot{\theta} > 0$, body rotates and translates.

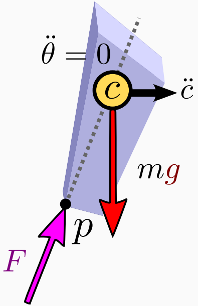


Figure 5: Net contact force goes through CoM c
$\Rightarrow \dot{L}_c = 0$, body translates but does not rotate.

Bottom line

A constant angular momentum reduces the system to translation

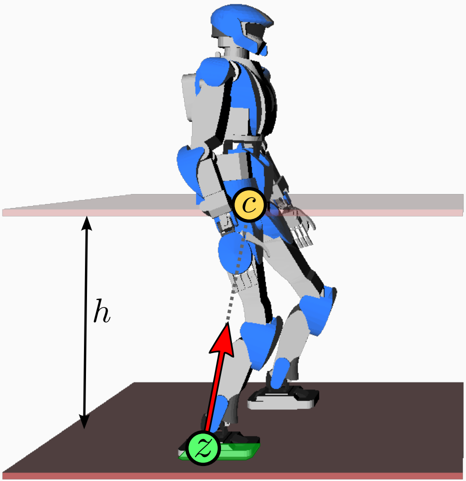
Assumptions:

- Rigid joints, sufficient power
- Conservation of angular momentum
- Constant CoM height

Equation of motion

$$\ddot{c} = \omega^2 (c - z) + g$$

- ω is a constant
- z: *zero-tilting moment point* (ZMP)
- In horiz. plane $+g$ is usually omitted



[6] Shuuji Kajita, Fumio Kanehiro, Kenji Kaneko, Kazuhito Yokoi, and Hirohisa Hirukawa. "The 3D Linear Inverted Pendulum Mode: A simple modeling for a biped walking pattern generation". In: *IEEE/RSJ International Conference on Intelligent Robots and Systems*. 2001.

Calculate the LIP constant

Equations and assumptions we have seen/made so far:

- Centroidal dynamics:

$$\ddot{c} = g + \frac{1}{m} \sum_{i \in \text{contacts}} f_i \qquad \dot{L}_c = \sum_{i \in \text{contacts}} (p_i - c) \times f_i$$

- Zero-tilting moment point:

$$\tau_z \times e_z = \left[\sum_{i \in \text{contacts}} (p_i - z) \times f_i \right] \times e_z = 0$$

- Constant height: $c \cdot e_z = h$
- Angular momentum: $\dot{L}_c = 0$

Question

Derive the LIP equation $\ddot{c} = \omega^2 (c - z) + g$. What is the expression of ω?

The ZMP should lie in a support area:

$$z_{\text{min}} \leq z \leq z_{\text{max}}$$

If not, the surface contact will break into a line or a point contact.

On a flat floor and under large friction, the ZMP support area is simply the convex hull of ground contact points.[7]

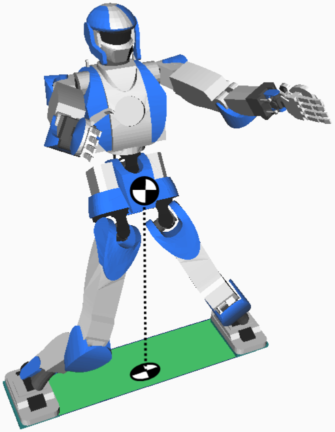


Figure 6: ZMP support area under large floor friction.

[7] This construction does not generalize to arbitrary contacts.

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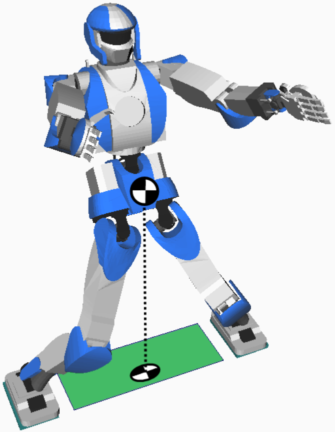


Figure 6: ZMP support area under limited floor friction.

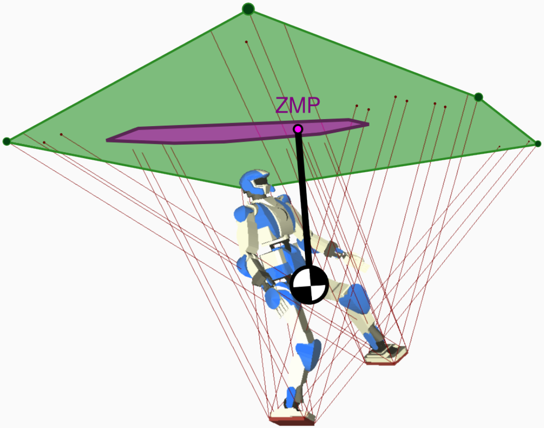
[7] This construction does not generalize to arbitrary contacts.

ZMP support area

The ZMP support area can always be computed by polyhedral geometry:

$$Gz \leq h$$

where G and h depend on the location, geometry and friction of contact surfaces.



[8] Stéphane Caron, Quang-Cuong Pham, and Yoshihiko Nakamura. "ZMP Support Areas for Multi-contact Mobility Under Frictional Constraints". In: *IEEE Transactions on Robotics* 33.1 (2017).

Quadratic programming

Quadratic programming

A quadratic program can be generally written as:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \frac{1}{2} x^T P x + q^T x \\ \text{subject to} \quad & Gx \leq h \\ & Ax = b \\ & lb \leq x \leq ub \end{aligned}$$

In Python:

```python
from qpsolvers import solve_qp

M = np.array([[1., 2., 0.], [-8., 3., 2.], [0., 1., 1.]])
P = M.T @ M  # this is a positive definite matrix
q = np.array([3., 2., 3.]) @ M
G = np.array([[1., 2., 1.], [2., 0., 1.], [-1., 2., -1.]])
h = np.array([3., 2., -2.])

x = solve_qp(P, q, G, h, solver="proxqp")
```

Setup: `pip install qpsolvers[open_source_solvers]`

Optimal control for walking

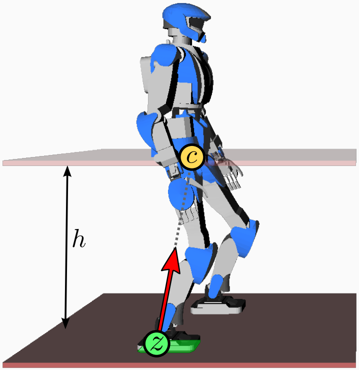
Control of a linear inverted pendulum

Continuous-time system:

- **Equation of motion:** $\ddot{c} = \omega^2 (c - z) + g$
- **Constraints:** $Gz \leq h$

Problems:

- **Controlability:** find $z(t)$ such that $c(t) \to c^*$
- **Optimal control:** find $z \in \arg\min_z \int_0^T \ell(c, \dot{c}, z) \mathrm{d}t$



Linear time-invariant system

- Discretize time into steps t_k such that $t_{k+1} = t_k + T$
- Notation: $v[k] = v(t_k)$ for any quantity v
- Let's decide $\dddot{c}$ is piecewise constant:

$$
\begin{aligned}
c[k+1] &= c[k] + T\dot{c}[k] + \frac{T^2}{2}\ddot{c}[k] + \frac{T^3}{6}\dddot{c}[k] \\
\dot{c}[k+1] &= \dot{c}[k] + T\ddot{c}[k] + \frac{T^2}{2}\dddot{c}[k] \\
\ddot{c}[k+1] &= \ddot{c}[k] + T\dddot{c}[k]
\end{aligned}
$$

- Define the **state** $x[k] := \begin{bmatrix} c[k] \\ \dot{c}[k] \\ \ddot{c}[k] \end{bmatrix}$ and **input** $u[k] := \begin{bmatrix} \dddot{c}[k] \end{bmatrix}$
- **Linear time-invariant system:** $x[k+1] = Ax[k] + Bu[k]$

Trajectory formulation

Looking at the first few steps:

$$x[1] = Ax[0] + Bu[0]$$
$$x[2] = A^2x[0] + [AB\ B][u[0]^T\ u[1]^T]^T$$
$$x[3] = A^3x[0] + [A^2B\ AB\ B][u[0]^T\ u[1]^T\ u[2]^T]^T$$

Define the state and input trajectory vectors:

$$X = \begin{bmatrix} x[0] \\ \vdots \\ x[N] \end{bmatrix} \qquad U = \begin{bmatrix} u[0] \\ \vdots \\ u[N-1] \end{bmatrix}$$

Then we have $X = \Phi x[0] + \Psi U$ with:

$$\Phi = \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} \qquad \Psi = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & B & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}$$

Trajectory constraints

Continuous-time system:

- **Dynamics:** $\ddot{c} = \omega^2 (c - z) + g$
- **Constraints:** $G z \leq h$

We can write equivalently:

- **Dynamics:** $z[k] = c[k] - \frac{\ddot{c}[k]}{\omega^2} = Z_k X = Z_k \Phi x[0] + Z_k \Psi U$
- **Feasibility:** $z_{\text{min}}[k] \leq z[k] \leq z_{\text{max}}[k]$

Our whole trajectory is linearly defined in U!

Continuous-time system:

- **Equation of motion:** $\ddot{c} = \omega^2(c - z) + g$
- **Constraints:** $Gz \leq h$

Problems:

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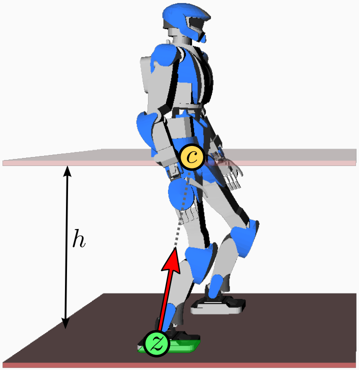

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```

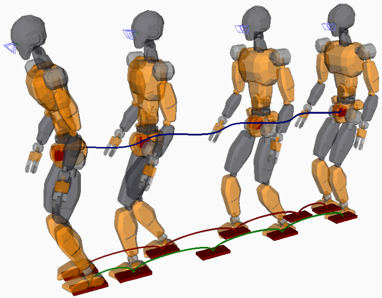
Setup: `pip install qpsolvers[open_source_solvers]`

Cost function

- Track desired ZMP reference
- Track desired CoM velocity
- Minimize CoM jerk

Constraints

- Transition: dynamics
- Feasibility: ZMP in support area
- (Viability: terminal DCM)



[9] Pierre-Brice Wieber. "Trajectory free linear model predictive control for stable walking in the presence of strong perturbations". In: *IEEE-RAS International Conference on Humanoid Robots*. 2006.

Linear OCP is a quadratic program

$$\min_{U} \quad w_z \sum_{k=1}^{N} \|z[k] - z^d[k]\|^2 + w_v \sum_{k=1}^{N} \|\dot{c}[k] - \dot{c}^d[k]\|^2 + w_j \sum_{k=1}^{N} \|u[k]\|^2$$

$$\text{s.t. } \forall k \quad c[k+1] = c[k] + T\dot{c}[k] + \frac{T^2}{2}\ddot{c}[k] + \frac{T^3}{6}u[k]$$

$$\dot{c}[k+1] = \dot{c}[k] + T\ddot{c}[k] + \frac{T^2}{2}u[k]$$

$$\ddot{c}[k+1] = \ddot{c}[k] + Tu[k]$$

Dynamics: $z[k] = c[k] - \frac{\ddot{c}[k]}{\omega^2}$

Feasibility: $z_{\text{min}}[k] \leq z[k] \leq z_{\text{max}}[k]$

$$(\text{Viability: } c[N] + \frac{\dot{c}[N]}{\omega} = \xi^d[N])$$

[10] Pierre-Brice Wieber. "Trajectory free linear model predictive control for stable walking in the presence of strong perturbations". In: *IEEE-RAS International Conference on Humanoid Robots*. 2006.

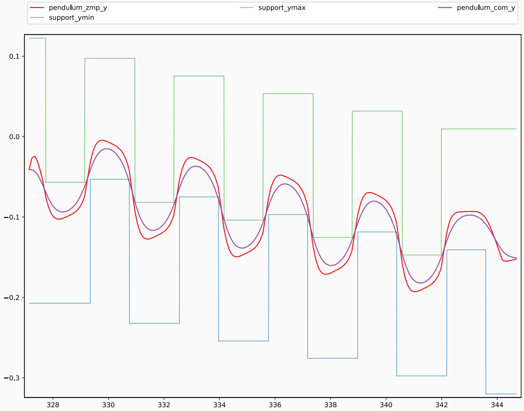


Figure 7: Horizontal axis for time, vertical axis for lateral coordinates $c_y, z_y, z_{\text{min}}, z_{\text{max}}$.

Application in a real-world environment

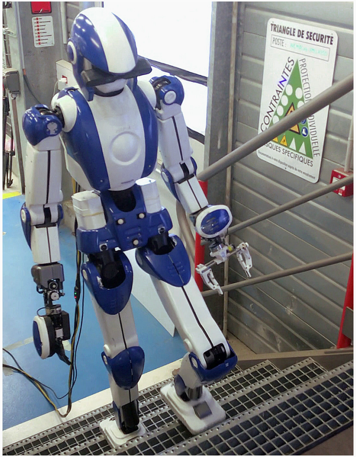


Figure 8: LIPM walking controller

Code: `https://github.com/stephane-caron/lipm_walking_controller` (2019)

Lateral walking plan and measurements

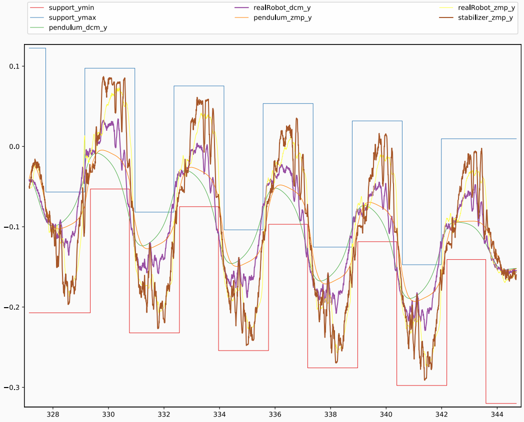
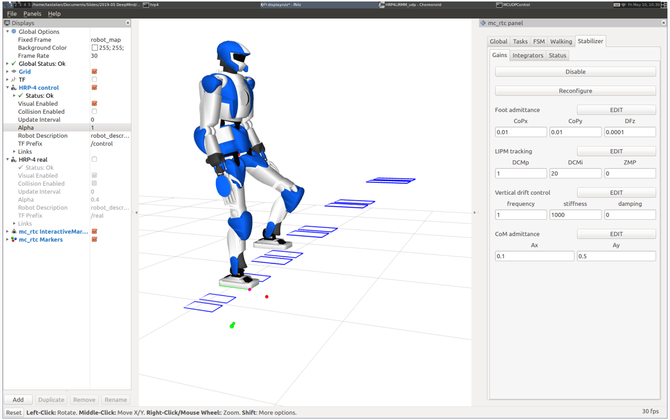


Figure 9: Horizontal axis for time, vertical axis for lateral coordinates $c_y, z_y, z_{\text{min}}, z_{\text{max}}$.

Try it out in a Docker!

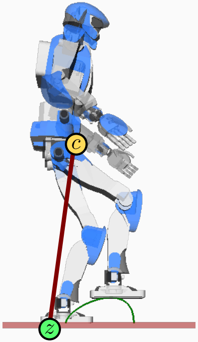



```
$ xhost +  # for X11 forwarding
$ docker run -it --rm --user ayumi -e DISPLAY=${DISPLAY} \
    -v /tmp/.X11-unix:/tmp/.X11-unix:rw \
    stephanecaron/lipm_walking_controller \
    lipm_walking --staircase
```

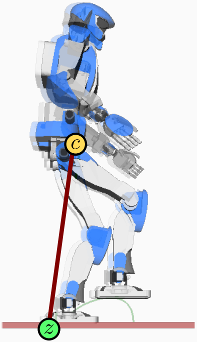
Figure 10: Humanoid, centroidal dynamics

Figure 11: Quadruped, centroidal dynamics

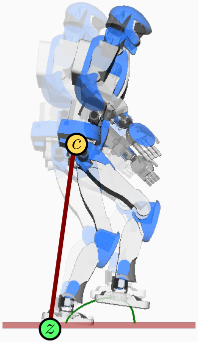
Balance control



Plan$(t + \Delta t)$



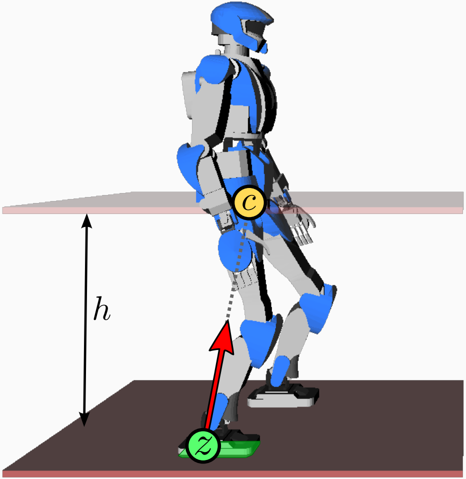
Simulation$(t + \Delta t)$



Real$(t + \Delta t)$

Assumptions:

- Rigid joints, sufficient power
- Conservation of angular momentum
- Constant CoM height



Equation of motion

$$\ddot{c} = \omega^2 (c - z) + g$$

- ω is a constant
- z: *zero-tilting moment point* (ZMP)
- In horiz. plane $+g$ is usually omitted

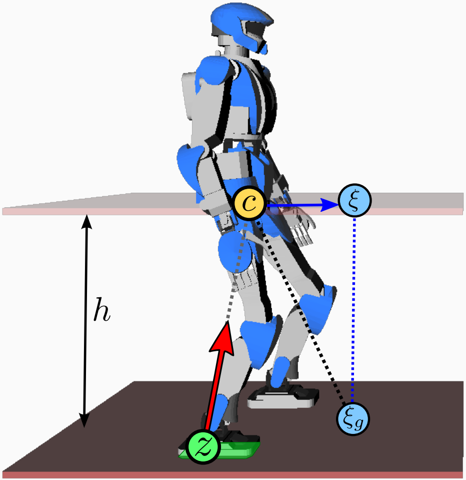
[11] Shuuji Kajita, Fumio Kanehiro, Kenji Kaneko, Kazuhito Yokoi, and Hirohisa Hirukawa. "The 3D Linear Inverted Pendulum Mode: A simple modeling for a biped walking pattern generation". In: *IEEE/RSJ International Conference on Intelligent Robots and Systems*. 2001.

- Linear inverted pendulum mode: $\ddot{c} = \omega^2(c - p)$
- Divergent component of motion: $\xi := c + \frac{\dot{c}}{\omega}$

DCM dynamics

$$\dot{\xi} = \omega(\xi - p)$$

- Point ξ_g where stepping will stop walking



Balance control with feet

- DCM dynamics:

$$\dot{\xi} = \omega(\xi - z)$$

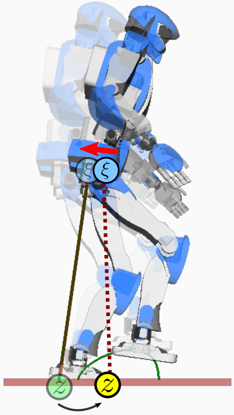
- Regulate the ZMP by **force control**:

$$z = z^d + \xi - k(\xi^d - \xi)$$

- Closed loop: $\xi \rightarrow \xi^d$

$$\dot{\xi} = k\omega(\xi^d - \xi)$$

- As long as the ZMP target is feasible...



Balance control with wheels

- DCM dynamics:

$$\dot{\xi} = \omega(\xi - z)$$

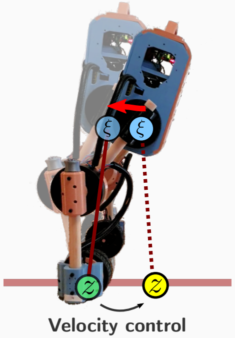
- Regulate the ZMP by **velocity control**:

$$z = z^d + \xi - k(\xi^d - \xi)$$

- Closed loop: $\xi \rightarrow \xi^d$

$$\dot{\xi} = k\omega(\xi^d - \xi)$$

- As long as the ZMP target is feasible...



What did we see?

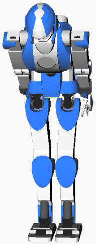
What did we see?

Balancing and walking legged robots:

- **Physics:** linear inverted pendulum, ZMP, support area
- **Optimal control:** LTI system, OC as a quadratic program
- **Balancing:** adjust ZMP based on a DCM of the CoM

Applies to humanoids, quadrupeds, wheeled bipeds, flywheeled coffee makers, etc.

That's all folks!



Bibliography

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