

# Kinematics and rigid transformations

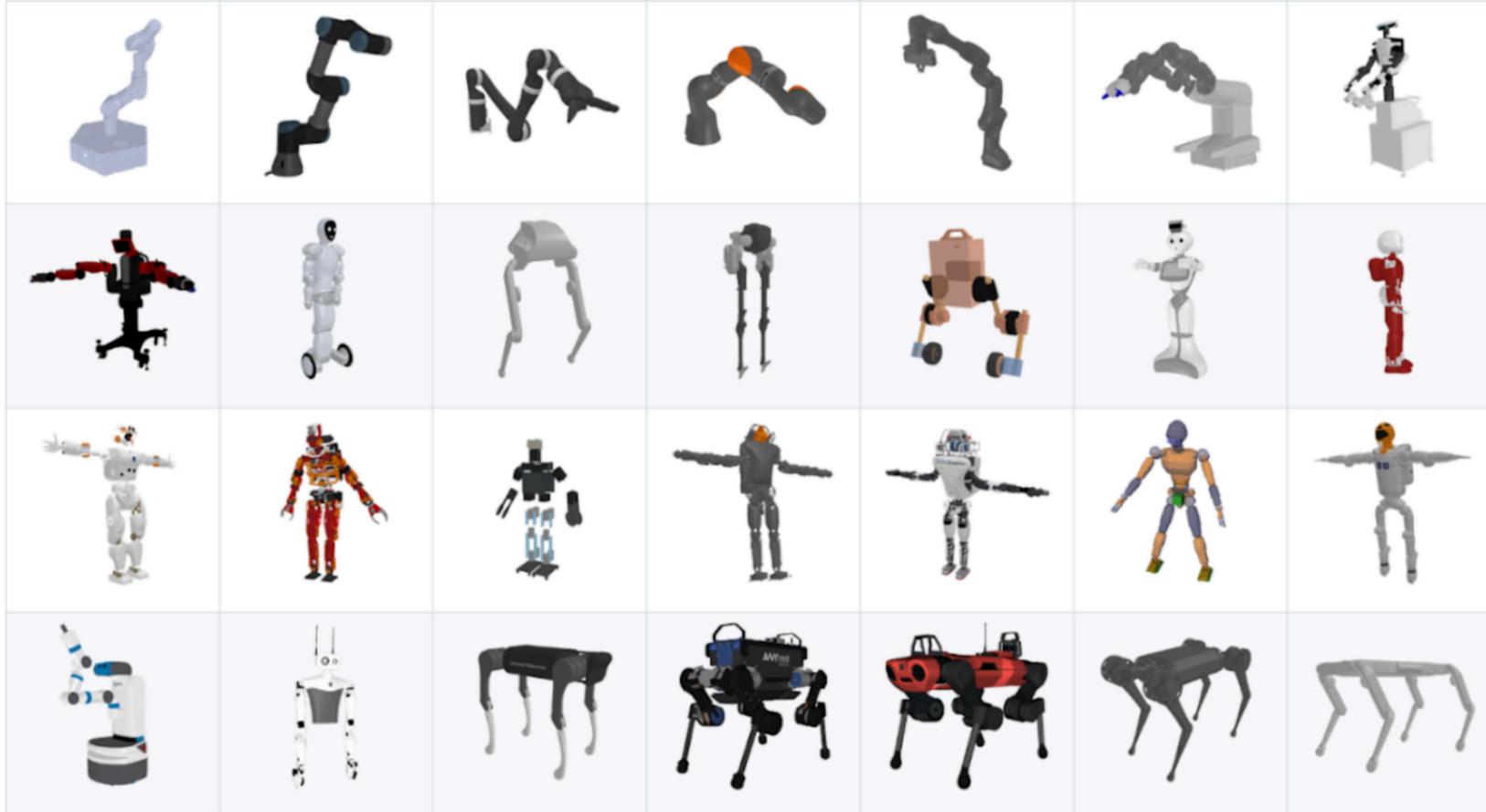
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Robotics - Master MVA

16 October 2025

## Next lectures

Date	Where	Topic
23/10/25	L108_B, Ecole des Mines	Inverse kinematics
30/10/25	L109, Ecole des Mines	Perception and estimation
06/11/25	Salle Jaurès, ENS Ulm	Motion planning
13/11/25	L109, Ecole des Mines	Optimal control and simulation
20/11/25	1Z71, ENS Paris-Saclay	Reinforcement learning for legged robots
27/11/25	1Z71, ENS Paris-Saclay	Responsible robotics
04/12/25	–	–
11/12/25	Inria Paris	Final poster session



# Kinematics

Kinematics considers *motion* of points and rigid bodies:

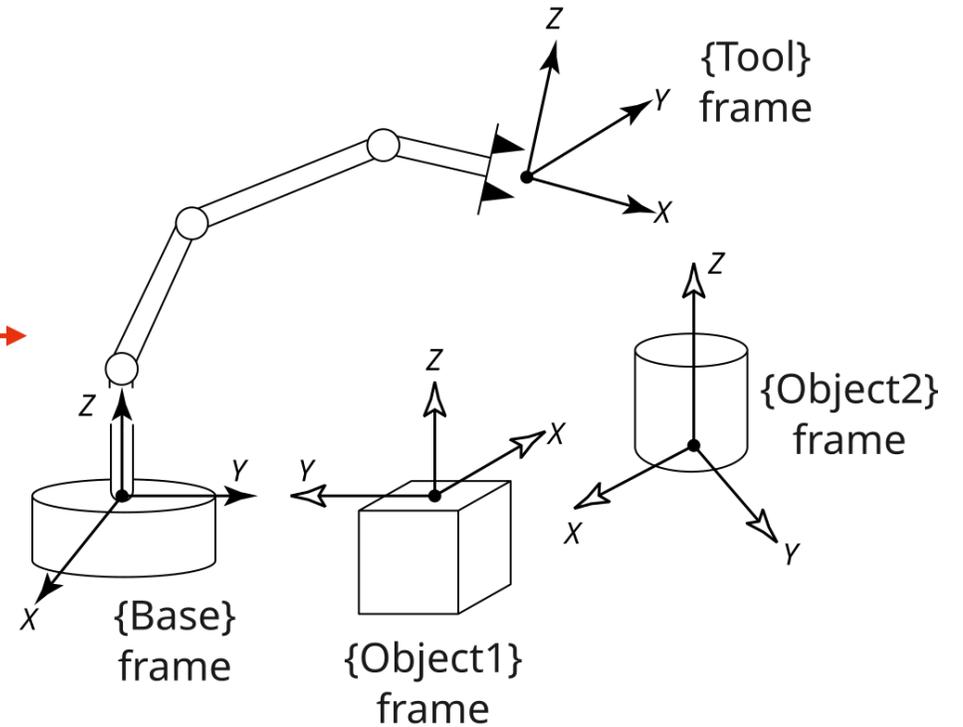
- Positions
- Orientations
- Linear velocities
- Angular velocities
- etc.

(Forces that cause motion will be the object of *dynamics*.)

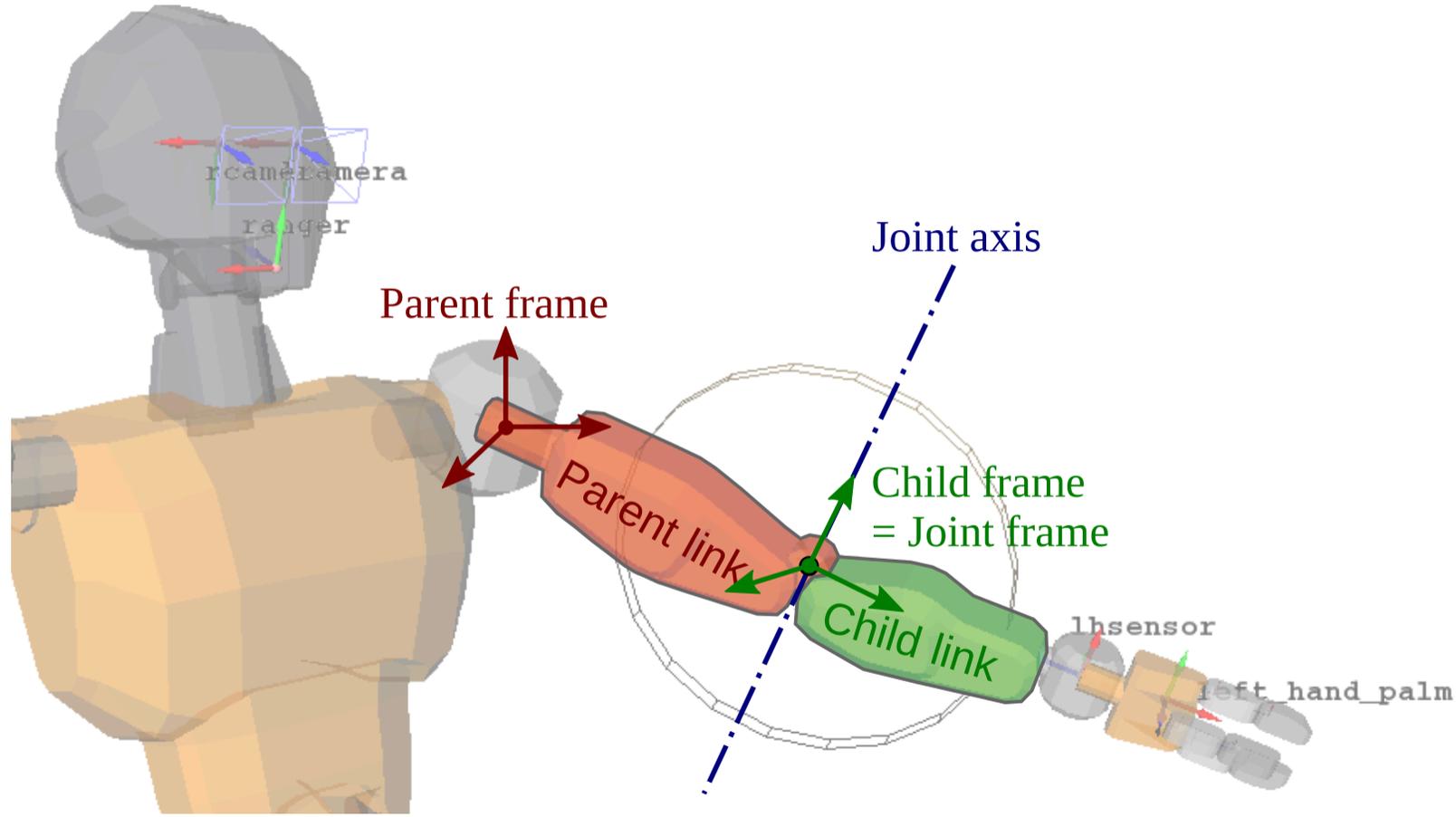
# Frames attached to rigid bodies



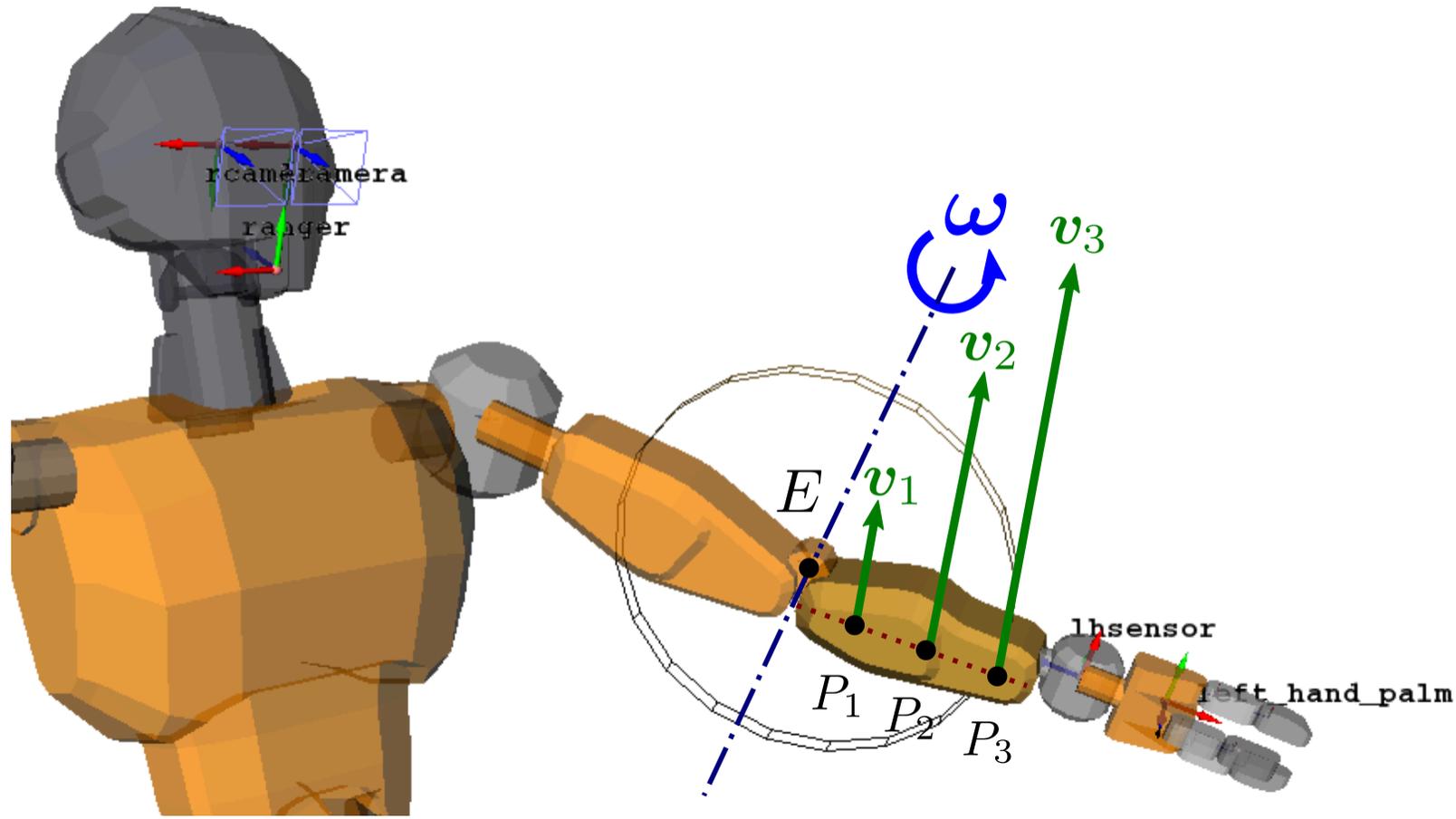
Model the robot  
and its environment



# Links and joints



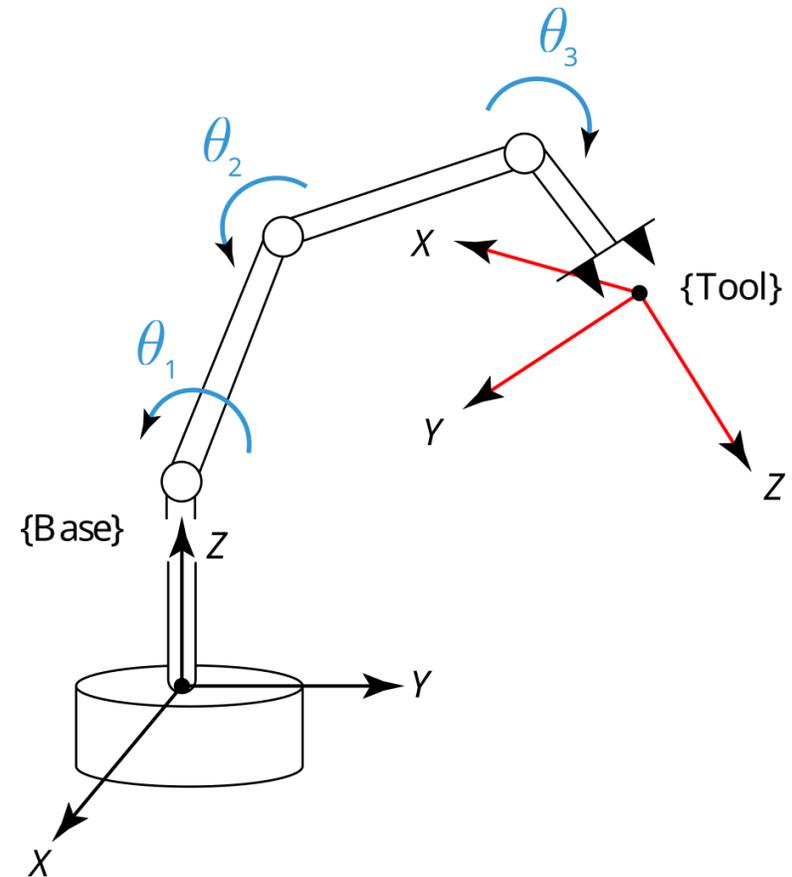
# Rotation of a revolute joint



## Forward kinematics

Given the **angles** of each revolute joint, how to calculate the **position and orientation** of the end-effector frame?

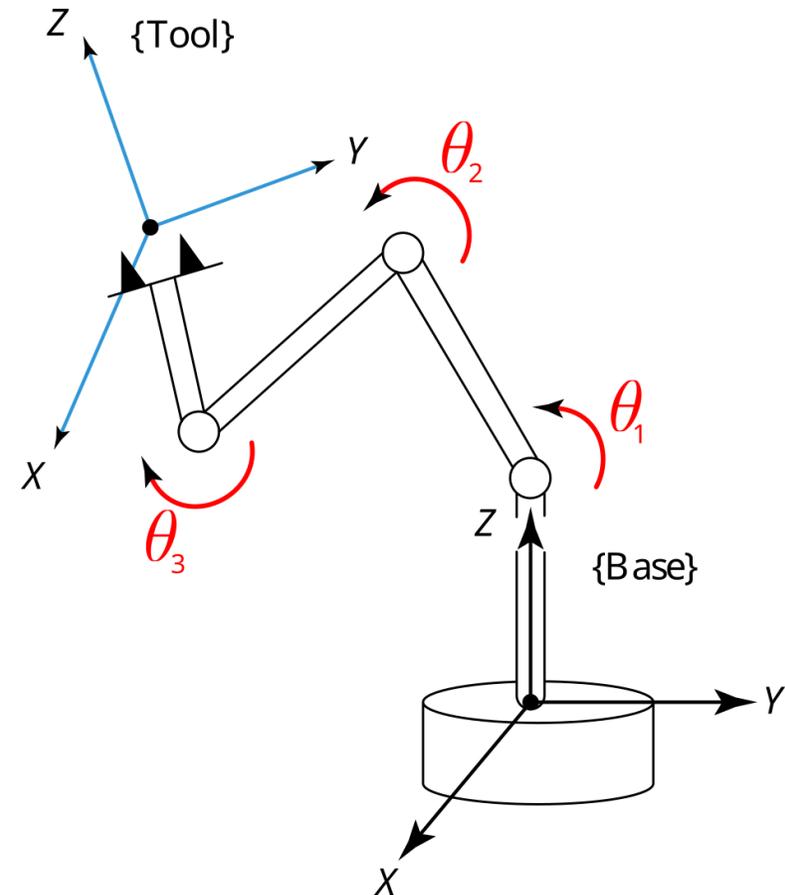
This is **forward kinematics**, a fundamental problem in robotics.



## Inverse kinematics

Given a target **position and orientation**,  
how to calculate **joint angles** that put the end  
effector in that pose?

This is **inverse kinematics**, a key problem to  
robot control and planning.



# Outline

1.  $SO(2)$ : Rotations in 2D space
2.  $SE(2)$ : Rigid transformations in 2D space
3.  $SO(3)$ : Rotations in 3D space
4.  $SE(3)$ : Rigid transformations in 3D space
5. ★ Error log-linearity property

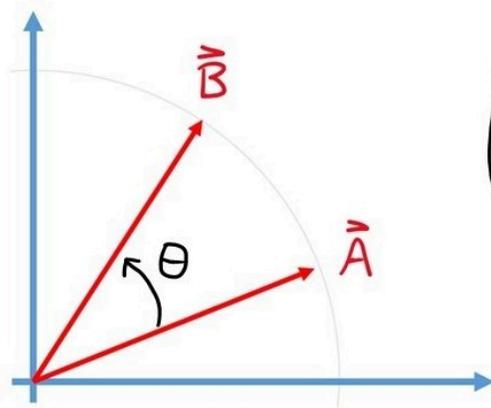
# The group $SO(2)$

## Definition:

The set  $SO(2)$  denotes the group of rotations in  $\mathbb{R}^2$ .

The rotation of angle  $\theta \in \mathbb{R}$  can be represented by the rotation matrix:

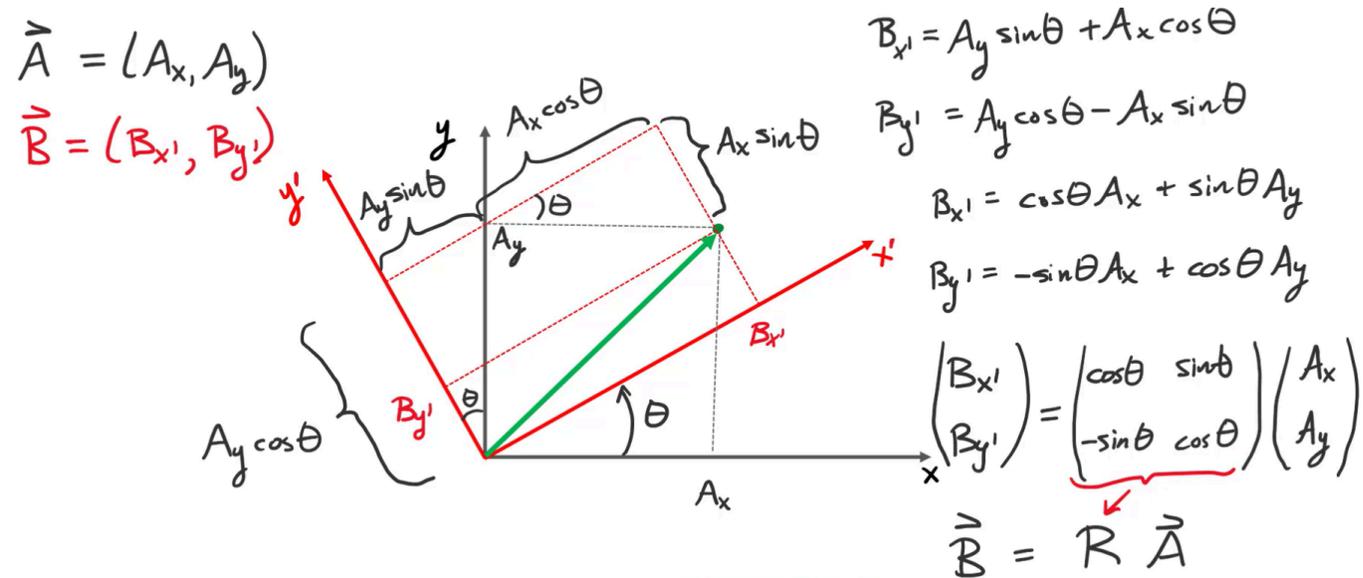
$$\mathbf{R}(\theta) := \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$



$$\begin{pmatrix} B_x \\ B_y \end{pmatrix} = \underbrace{\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}}_{\mathbf{R}(\theta) \Rightarrow \text{rotation matrix}} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

# The group $SO(2)$ : the change-of-frame viewpoint

From our perspective,  $\mathbf{R}(\theta)$  encodes a change of frame:



## Take-home message

$\mathbf{R}(\theta)$  maps the coordinates of a vector expressed in one frame to the coordinates of the same vector in another (rotated) frame.

## The group $SO(2)$

### Theorem

$SO(2)$  is a group. Notably we have:

$$\mathbf{R}(\theta_1)\mathbf{R}(\theta_2) = \mathbf{R}(\theta_1 + \theta_2)$$

and:

$$\mathbf{R}(\theta)^{-1} = \mathbf{R}(-\theta)$$

## The group $SO(2)$ : how to differentiate

Consider a continuous curve  $\theta(t)$  defined by  $\frac{d}{dt}\theta(t) = \omega(t)$ , with  $\omega(t)$  the angular velocity.

### Theorem

The corresponding curve on  $SO(2)$  writes:

$$\frac{d}{dt}\mathbf{R}(\theta) = \omega\mathbf{J}\mathbf{R}(\theta)$$

where  $\mathbf{J} := \mathbf{R}(\pi/2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .

Proof:

$$\frac{d}{dt}\mathbf{R}(\theta) = \begin{bmatrix} -\omega \sin \theta & -\omega \cos \theta \\ \omega \cos \theta & -\omega \sin \theta \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

## The group $SO(2)$ : the exponential map

Being able to differentiate makes it a Lie group. Then, an important concept is that of *exponential map*.

Assume you fix the angular velocity  $\omega(t) \equiv \bar{\omega} \in \mathbb{R}$ . Consider the ODE:

$$\mathbf{R}_0 = \mathbf{Id}_2, \quad \frac{d}{dt} \mathbf{R}_t = \bar{\omega} \mathbf{J} \mathbf{R}_t$$

### Theorem

The solution writes at all times  $t \geq 0$ :

$$\mathbf{R}_t = \exp \begin{bmatrix} 0 & -\bar{\omega}t \\ \bar{\omega}t & 0 \end{bmatrix} = \exp(\bar{\omega} \mathbf{J} t)$$

with  $\exp(\mathbf{M}) := \mathbf{Id}_2 + \mathbf{M} + \frac{1}{2!} \mathbf{M}^2 + \frac{1}{3!} \mathbf{M}^3 + \dots$  the matrix exponential.

## The group $SO(2)$ : the exponential map

### Definition

The exponential map of the group is the solution to the ODE  $\mathbf{R}_0 = \mathbf{Id}_2$ ,  $\frac{d}{dt}\mathbf{R}_t = \bar{\omega}\mathbf{J}\mathbf{R}_t$  at time  $t = 1$ . It is a map:

$$\begin{aligned}\exp_{SO(2)} : \mathbb{R} &\longrightarrow SO(2) \\ \bar{\omega} &\longmapsto \mathbf{R}_1\end{aligned}$$

Building on the previous theorem with  $t = 1$ , it writes:

$$\exp_{SO(2)}(\omega) = \exp(\omega\mathbf{J})$$

where  $\exp(\omega\mathbf{J}) := \mathbf{Id}_2 + \omega\mathbf{J} + \frac{1}{2!}(\omega\mathbf{J})^2 + \frac{1}{3!}(\omega\mathbf{J})^3 + \dots$  denotes the matrix exponential of  $\omega\mathbf{J}$ .

In fact,  $\exp_{SO(2)}(\alpha) = \mathbf{R}(\alpha)$  as we recover the Taylor expansions of sine and cosine.

## The group $SO(2)$ : the exponential map

### Take-home message

We take an angular velocity  $\omega$  and we define  $\exp_{SO(2)}$  through:

$$\exp_{SO(2)} : \omega \mapsto \omega \mathbf{J} \mapsto \exp(\omega \mathbf{J})$$

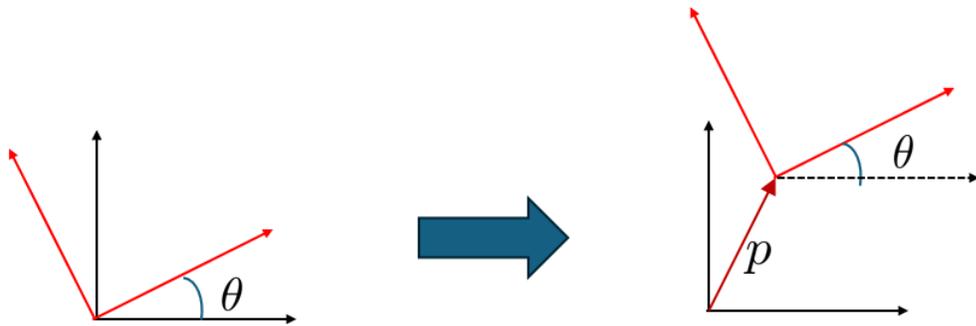
# Outline

1.  $SO(2)$ : Rotations in 2D space
2.  $SE(2)$ : Rigid transformations in 2D space
3.  $SO(3)$ : Rotations in 3D space
4.  $SE(3)$ : Rigid transformations in 3D space
5. ★ Error log-linearity property

# The group $SE(2)$

## Definition

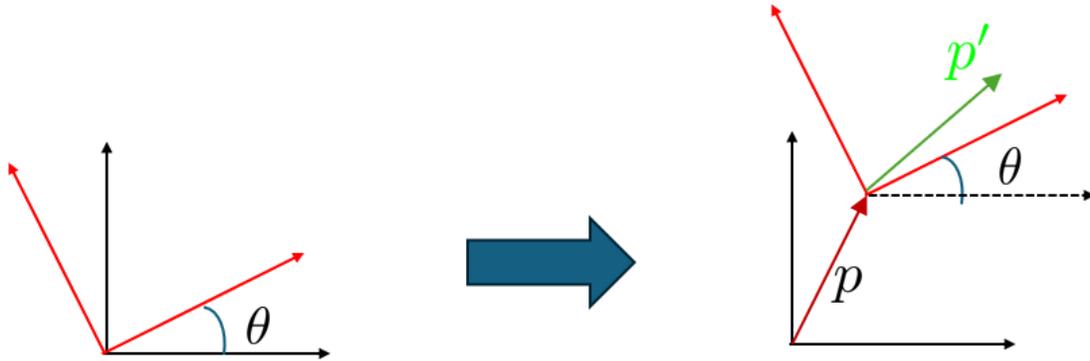
The set  $SE(2)$  denotes the group of rigid transformations, *i.e.* rotation + translation, in  $\mathbb{R}^2$ .



# The group $SE(2)$ : the change of frame viewpoint

## Definition

The set  $SE(2)$  denotes the group of rigid transformations, *i.e.* rotation + translation, in  $\mathbb{R}^2$ .



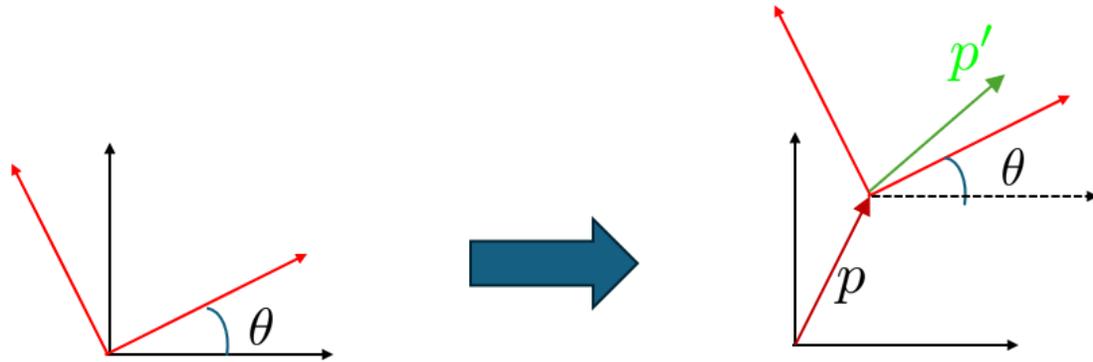
If  $\mathbf{p}'$  denotes a point expressed in the rotated and translated frame, the coordinates of this point in the initial frame are  $\mathbf{R}(\theta)\mathbf{p}' + \mathbf{p}$ .

*Example:* if  $\mathbf{p}' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ , the corresponding point expressed in the initial frame is  $\mathbf{p}$ , which is indeed equal to  $\mathbf{R}(\theta)\mathbf{p}' + \mathbf{p}$ .

# The group $SE(2)$ : the change of frame viewpoint

## Definition

The set  $SE(2)$  denotes the group of rigid transformations, *i.e.* rotation + translation, in  $\mathbb{R}^2$ .



## Take-home message

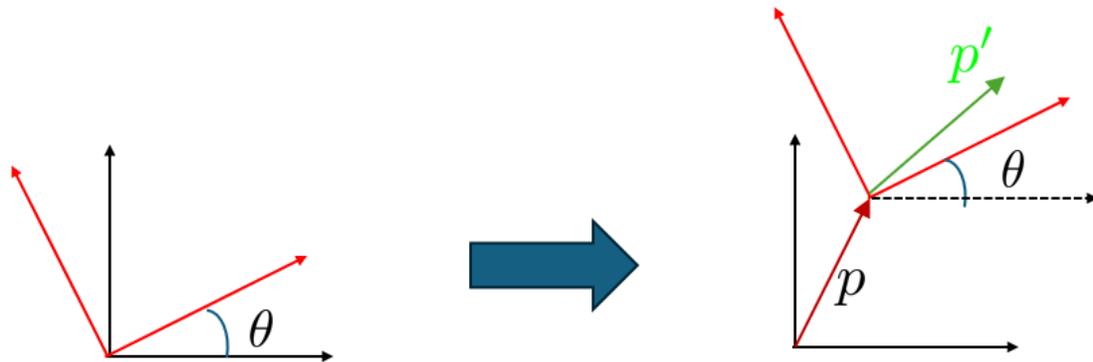
The rigid transformation maps the coordinates of a vector expressed in the red frame to the coordinates of the same vector in the black original frame.

# The group $SE(2)$ : matrix representation

## Definition

The rigid transformation of angle  $\theta \in \mathbb{R}$  and translation vector  $p$  can be represented by the matrix:

$$\chi := \begin{bmatrix} \mathbf{R}(\theta) & \mathbf{p} \\ 0 & 1 \end{bmatrix} := \begin{bmatrix} \cos \theta & -\sin \theta & p_x \\ \sin \theta & \cos \theta & p_y \\ 0 & 0 & 1 \end{bmatrix}$$



We see indeed that  $\mathbb{R}^3 \ni \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} R(\theta) & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p' \\ 1 \end{bmatrix}$ .

## The group $SE(2)$

### Theorem

$SE(2)$  is a group, letting the group law stem from matrix multiplication:

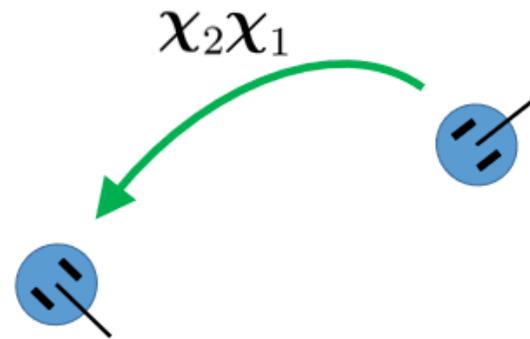
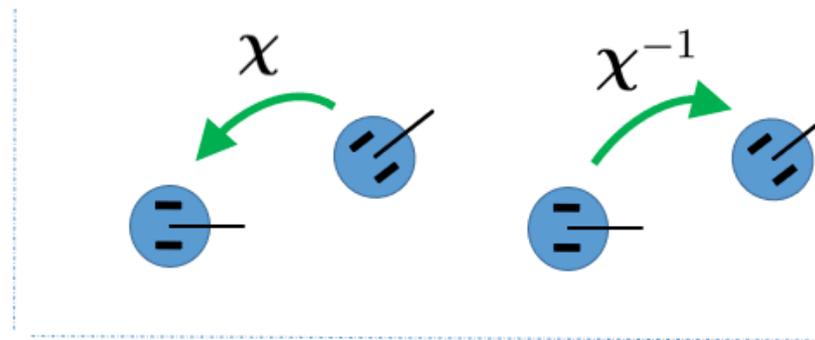
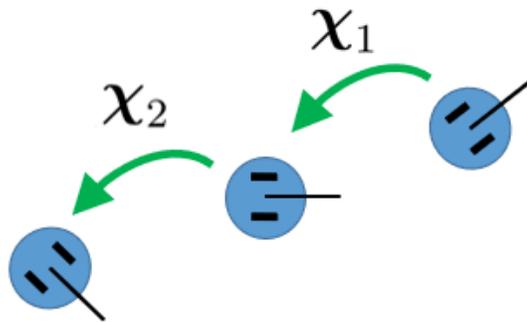
$$\boldsymbol{\chi}_1 := \begin{bmatrix} \mathbf{R}(\theta_1) & \mathbf{p}_1 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{\chi}_2 := \begin{bmatrix} \mathbf{R}(\theta_2) & \mathbf{p}_2 \\ 0 & 1 \end{bmatrix}$$

- Two successive changes of frames correspond to the transformation generated by  $\boldsymbol{\chi}_1\boldsymbol{\chi}_2$
- The inverse transformation is represented by the matrix  $\boldsymbol{\chi}^{-1}$

## The group $SE(2)$ : illustration

$$\chi := \begin{pmatrix} \cos \theta & -\sin \theta & x \\ \sin \theta & \cos \theta & y \\ 0 & 0 & 1 \end{pmatrix}$$

- Matrix **group**



## The group $SE(2)$ : how to differentiate

- Consider a continuous curve of velocities  $(\omega(t), \mathbf{v}(t))$  with  $\omega(t) \in \mathbb{R}$  and  $\mathbf{v}(t) \in \mathbb{R}^2$ . The whole point is to couple evolutions of corresponding  $(\theta(t), \mathbf{p}(t))$  so as to "stick" to some systems.
- We define a curve on  $SE(2)$  analogously to what we did with rotation matrices:

$$\frac{d}{dt}\chi_t = \chi_t \Omega_t, \quad \Omega_t := \begin{bmatrix} \omega(t)\mathbf{J} & \mathbf{v}(t) \\ 0 & 0 \end{bmatrix}$$

### Theorem

The curve thus defined, with matrix representation  $\chi_t$ , satisfies:

$$\begin{aligned} \frac{d}{dt}\mathbf{p}(t) &= \mathbf{R}(\theta(t))\mathbf{v}(t) \\ \frac{d}{dt}\theta(t) &= \omega(t) \end{aligned}$$

*Interpretation:*  $\mathbf{v}$  is the velocity vector in the moving frame.

## The group $SE(2)$ : the exponential map

Being able to differentiate makes it a Lie group. Then, an important concept is that of exponential map.

Assume you fix the velocities  $\omega(t) \equiv \bar{\omega} \in \mathbb{R}$  and  $\mathbf{v}(t) \equiv \bar{\mathbf{v}} \in \mathbb{R}^2$ . Consider the following ODE:

$$\boldsymbol{\chi}_0 = \mathbf{I}_3, \quad \frac{d}{dt} \boldsymbol{\chi}_t = \boldsymbol{\chi}_t \bar{\boldsymbol{\Omega}}, \quad \bar{\boldsymbol{\Omega}} := \begin{bmatrix} \bar{\omega} \mathbf{J} & \bar{\mathbf{v}} \\ 0 & 0 \end{bmatrix}$$

### Theorem

The solution writes at all times  $t \geq 0$

$$\boldsymbol{\chi}_t = \exp(\bar{\boldsymbol{\Omega}}t)$$

with  $\exp(\mathbf{M}) := \mathbf{Id}_2 + \mathbf{M} + \frac{1}{2!} \mathbf{M}^2 + \frac{1}{3!} \mathbf{M}^3 + \dots$  the matrix exponential.

# The group $SE(2)$ : the exponential map

## Definition

The exponential map of the group is defined through matrix representations and the ODE

$\chi_0 = I_3$ ,  $\frac{d}{dt} \chi_t = \chi_t \bar{\Omega}$ , at time  $t = 1$ :

$$\exp_{SE(2)} : \mathbb{R}^3 \longrightarrow SE(2)$$

$$\begin{bmatrix} \bar{\omega} \\ \bar{\mathbf{v}} \end{bmatrix} \longmapsto \chi_1$$

Building on the previous theorem (let  $t = 1$ ) it writes:

$$\exp_{SE(2)} \begin{bmatrix} \omega \\ \mathbf{v} \end{bmatrix} = \exp(\Omega)$$

where  $\exp$  denotes the matrix exponential and  $\Omega := \begin{bmatrix} \omega \mathbf{J} & \mathbf{v} \\ 0 & 0 \end{bmatrix}$ .

# The group $SE(2)$ : the exponential map

Building on the previous theorem (let  $t = 1$ ) it writes

$$\exp_{SE(2)} \begin{bmatrix} \omega \\ \mathbf{v} \end{bmatrix} = \exp \begin{bmatrix} \omega \mathbf{J} & \mathbf{v} \\ 0 & 0 \end{bmatrix} = I_3 + \begin{bmatrix} \omega \mathbf{J} & \mathbf{v} \\ 0 & 0 \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} (\omega \mathbf{J})^2 & \omega \mathbf{J} \mathbf{v} \\ 0 & 0 \end{bmatrix} + \dots$$

This shows:

## Theorem

$$\exp \begin{bmatrix} \omega \mathbf{J} & \mathbf{v} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{R}(\omega) & \mathbf{V}(\omega) \mathbf{v} \\ 0 & 1 \end{bmatrix}$$

$$\text{with } \mathbf{V}(\omega) = \mathbf{I}_2 + \frac{1}{2!} (\omega \mathbf{J}) + \frac{1}{3!} (\omega \mathbf{J})^2 + \dots$$

We easily show the important equality

$$\mathbf{I}_2 + \omega \mathbf{J} \mathbf{V}(\omega) = \mathbf{I}_2 + \omega \mathbf{J} + \frac{1}{2!} (\omega \mathbf{J})^2 + \frac{1}{3!} (\omega \mathbf{J})^3 + \dots = \exp(\omega \mathbf{J}) = \mathbf{R}(\omega)$$

## Summary

- A rotation (resp. rigid transformation) is parameterized by an angle (resp. angle+translation vector). One may represent them by a matrix with a particular (nonlinear) structure.
- $SO(2)$  and  $SE(2)$  are groups. The group law corresponds to matrix *multiplication* (and inversion to inversion).
- The exponential map of the group maps a vector space ( $\mathbb{R}$  or  $\mathbb{R} \times \mathbb{R}^2$ ) to a matrix in the group. It is defined as the solution at  $t = 1$  of a matrix differential equation.
- It is related to the matrix exponential, and we have closed-form formulas for its expression.

### Take-home message

The exponential map relates any element of the group to a vector. It allows the application of methods tailored for vector spaces to Lie groups.

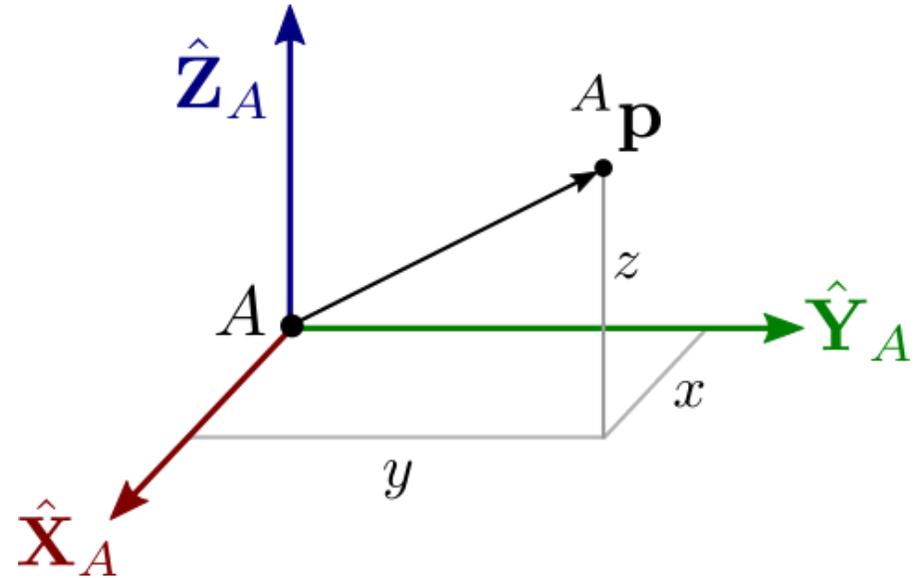
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5. ★ Error log-linearity property

## Position of a point in space

Once an *inertial* frame  $A$  is defined, we can locate any point in space by specifying its **position vector** in  $A$ :

$${}^A\mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



## Position and orientation of a body in space

For the orientation of a body in space, we attach another frame to it. Orientation is then defined as the expression of the body's coordinate system in that of the inertial frame:

$${}^A_B\mathbf{R} = \begin{bmatrix} {}^A\hat{\mathbf{X}}_B & {}^A\hat{\mathbf{Y}}_B & {}^A\hat{\mathbf{Z}}_B \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- expression of the coordinate system in a reference frame, here transformation from {B} to {A}
- relative position and orientation between frames {A} and {B}

## Position and orientation of a body in space

$${}^A_B\mathbf{R} = \begin{bmatrix} {}^A\hat{\mathbf{X}}_B & {}^A\hat{\mathbf{Y}}_B & {}^A\hat{\mathbf{Z}}_B \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{X}}_B \cdot \hat{\mathbf{X}}_A & \hat{\mathbf{Y}}_B \cdot \hat{\mathbf{X}}_A & \hat{\mathbf{Z}}_B \cdot \hat{\mathbf{X}}_A \\ \hat{\mathbf{X}}_B \cdot \hat{\mathbf{Y}}_A & \hat{\mathbf{Y}}_B \cdot \hat{\mathbf{Y}}_A & \hat{\mathbf{Z}}_B \cdot \hat{\mathbf{Y}}_A \\ \hat{\mathbf{X}}_B \cdot \hat{\mathbf{Z}}_A & \hat{\mathbf{Y}}_B \cdot \hat{\mathbf{Z}}_A & \hat{\mathbf{Z}}_B \cdot \hat{\mathbf{Z}}_A \end{bmatrix}$$

Each coordinate corresponds to the scalar product between the base axes of the two frames.

Rows correspond to axes of frame {A} expressed in frame {B}

$${}^A_B\mathbf{R} = \begin{bmatrix} {}^B\hat{\mathbf{X}}_A^T \\ {}^B\hat{\mathbf{Y}}_A^T \\ {}^B\hat{\mathbf{Z}}_A^T \end{bmatrix}$$

The inverse of a rotation matrix is its transpose itself:

$${}^B_A\mathbf{R} = {}^A_B\mathbf{R}^T \quad \Rightarrow \quad {}^A_B\mathbf{R} = {}^B_A\mathbf{R}^{-1} = {}^B_A\mathbf{R}^T$$

## The group $SO(3)$

The group of rotations in dimension 3 is the *special orthogonal group*, denoted  $SO(3)$ .

$\mathbf{R}$  is an orthonormal matrix if and only if  $\mathbf{R}\mathbf{R}^T = \mathbf{Id}_3$  and  $\det(\mathbf{R}) = +1$ .

Denote by  $\mathbf{R} = [\hat{\mathbf{X}} \quad \hat{\mathbf{Y}} \quad \hat{\mathbf{Z}}]$  the columns of a rotation matrix:

$$\begin{aligned} \|\hat{\mathbf{X}}\| &= 1, & \|\hat{\mathbf{Y}}\| &= 1, & \|\hat{\mathbf{Z}}\| &= 1 \\ \hat{\mathbf{X}} \cdot \hat{\mathbf{Y}} &= 0, & \hat{\mathbf{X}} \cdot \hat{\mathbf{Z}} &= 0, & \hat{\mathbf{Y}} \cdot \hat{\mathbf{Z}} &= 0 \end{aligned}$$

Rotation matrices have 9 degrees of freedom, but there are 6 constraints. Thus the group is 3-dimensional.

# Rotation representations

There are *many* ways to represent rotations. Each comes with advantages and disadvantages:

- **Orthonormal matrices:** 9 coordinates
- **Euler angles:** 3 coordinates, extrinsic or intrinsic, proper or Tait-Bryan (Best Confusers Award 🏆)
- **Angle-axis vectors:** 3 coordinates
- **Quaternions:** 4 coordinates

Enjoy the copious cheat sheets to switch from one to the other, for instance: 📖 [Representing attitude: Euler angles, unit quaternions, and rotation vectors.](#)

# Euler Angles

Sequence of 3 successive rotations around fixed axes, for instance X-Y-Z:

$${}^A_B \mathbf{R}_{XYZ}(\gamma, \beta, \alpha) = \mathbf{R}_Z(\alpha) \mathbf{R}_Y(\beta) \mathbf{R}_X(\gamma)$$

$$= \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \gamma & -\sin \gamma \\ 0 & \sin \gamma & \cos \gamma \end{bmatrix}$$

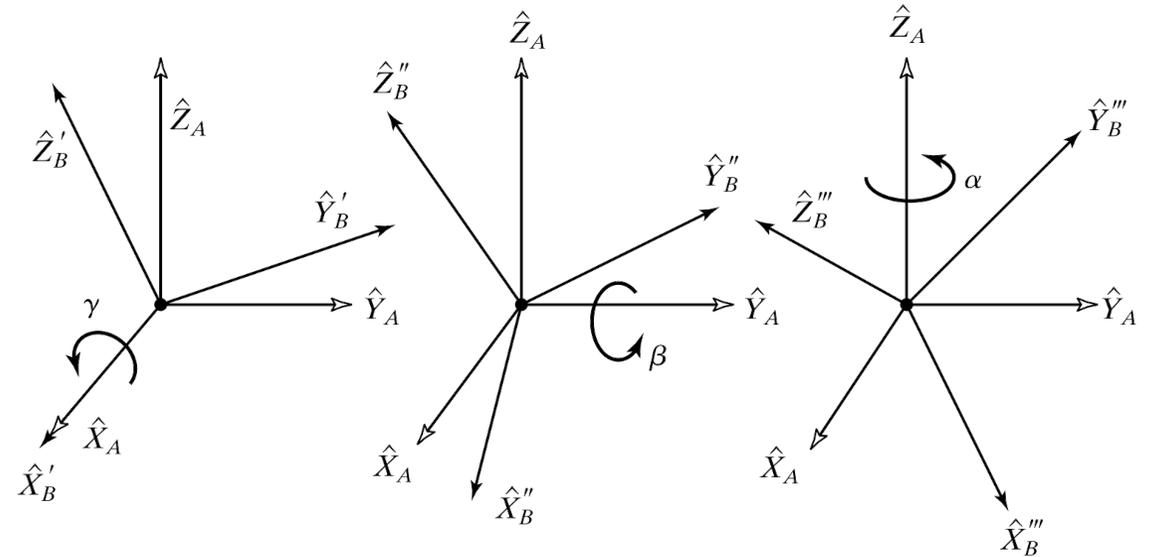
Inverse application:

$${}^A_B \mathbf{R}_{XYZ}(\gamma, \beta, \alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\beta = \text{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2})$$

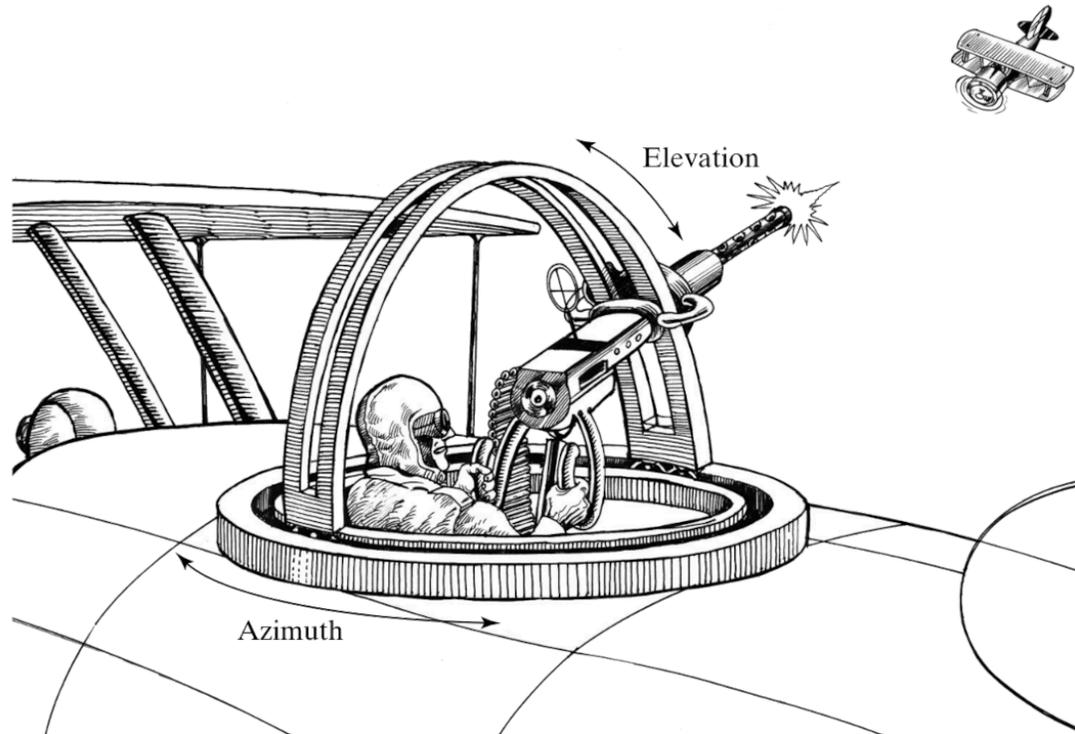
$$\alpha = \text{Atan2}(r_{21} / \cos \beta, r_{11} / \cos \beta)$$

$$\gamma = \text{Atan2}(r_{32} / \cos \beta, r_{33} / \cos \beta)$$



# Illustration of the concept of singularity

Euler angles suffer from **singularities** where the representation becomes ambiguous or undefined:



Classic example: when the elevation reaches  $\pm 90^\circ$ , azimuth becomes undefined.

## Angle axis and quaternion

A quaternion  $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$  consists of a unit vector  $\mathbf{k}$  and an angle  $\theta$ :

$$\epsilon_1 = k_x \sin \frac{\theta}{2}, \quad \epsilon_2 = k_y \sin \frac{\theta}{2}, \quad \epsilon_3 = k_z \sin \frac{\theta}{2}, \quad \epsilon_4 = \cos \frac{\theta}{2},$$

with the normalization constraint:  $\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 = 1$ .

The rotation matrix corresponding to this quaternion is:

$$\mathbf{R}_\epsilon = \begin{bmatrix} 1 - 2\epsilon_2^2 - 2\epsilon_3^2 & 2(\epsilon_1\epsilon_2 - \epsilon_3\epsilon_4) & 2(\epsilon_1\epsilon_3 + \epsilon_2\epsilon_4) \\ 2(\epsilon_1\epsilon_2 + \epsilon_3\epsilon_4) & 1 - 2\epsilon_1^2 - 2\epsilon_3^2 & 2(\epsilon_2\epsilon_3 - \epsilon_1\epsilon_4) \\ 2(\epsilon_1\epsilon_3 - \epsilon_2\epsilon_4) & 2(\epsilon_2\epsilon_3 + \epsilon_1\epsilon_4) & 1 - 2\epsilon_1^2 - 2\epsilon_2^2 \end{bmatrix}$$

The inverse application is:

$$\epsilon_1 = \frac{r_{32} - r_{23}}{4\epsilon_4} \quad \epsilon_2 = \frac{r_{13} - r_{31}}{4\epsilon_4} \quad \epsilon_3 = \frac{r_{21} - r_{12}}{4\epsilon_4} \quad \epsilon_4 = \frac{1}{2} \sqrt{1 + r_{11} + r_{22} + r_{33}}$$

## Angular velocity

Angular velocities correspond to the **time derivative of a rotation**. They lie on the **tangent space** to  $SO(3)$ , denoted  $so(3)$  or  $\mathfrak{so}(3)$ .

Consider a curve  $\mathbf{R}(t)$  in  $SO(3)$  parametrized by time:

$$\mathbf{R}(t)\mathbf{R}(t)^T = \mathbf{Id}_3$$

$$\dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{R}}^T = \mathbf{0}_3$$

$$\dot{\mathbf{R}}\mathbf{R}^T = -(\mathbf{R}\dot{\mathbf{R}}^T)^T$$

If we define  $\boldsymbol{\Omega} \stackrel{\text{def}}{=} -(\mathbf{R}\dot{\mathbf{R}}^T)^T$ , then  $\boldsymbol{\Omega} = -\boldsymbol{\Omega}^T$ .

## Group of skew-symmetric matrices

We have just seen that  $\Omega$  is a skew-symmetric matrix. It is therefore a member of the set:

$$\mathfrak{so}(3) \triangleq \left\{ \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}, (\omega_x, \omega_y, \omega_z) \in \mathbb{R}^3 \right\}$$

We can write  $\Omega = \hat{\omega}$  to highlight the vector  $\omega$  generating the matrix  $\Omega$ .

## Exponential map

Given:

$$\dot{\mathbf{R}}\mathbf{R}^T = -(\mathbf{R}\dot{\mathbf{R}}^T)^T$$

$$\mathbf{\Omega} = -\mathbf{\Omega}^T \text{ with } \mathbf{\Omega} \stackrel{\text{def}}{=} -(\mathbf{R}\dot{\mathbf{R}}^T)^T$$

$$\dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{\Omega} \text{ with initial conditions } \mathbf{R}(0) = \mathbf{R}_0 \in \text{SO}(3)$$

$$\mathbf{R}(t) = \mathbf{R}(0) \exp(t\mathbf{\Omega}) \text{ with } \exp(t\mathbf{\Omega}) = e^{t\mathbf{\Omega}} = \sum_{n=0}^{\infty} \frac{1}{n!} (t\mathbf{\Omega})^n$$

which gives us a new parameterization of rotations based on  $\boldsymbol{\omega} \in \mathbb{R}^3$  such that  $\mathbf{\Omega} = \hat{\boldsymbol{\omega}}$ .

The vector  $\boldsymbol{\omega}$  corresponds to the angular velocity expressed in the moving frame.

## Exponential map of $SO(3)$

### Definition

The exponential map of  $SO(3)$  is the solution to the ODE  $\mathbf{R}(0) = \mathbf{Id}_3$ ,  $\frac{d}{dt}\mathbf{R}(t) = \mathbf{R}(t)\mathbf{\Omega}$  at time  $t = 1$ . It is a map:

$$\begin{aligned}\exp_{SO(3)} : \mathfrak{so}(3) &\longrightarrow SO(3) \\ \mathbf{\Omega} &\longmapsto \mathbf{R}(1)\end{aligned}$$

We can express it via the matrix exponential:

$$\exp(t\mathbf{\Omega}) = e^{t\mathbf{\Omega}} = \sum_{n=0}^{\infty} \frac{1}{n!} (t\mathbf{\Omega})^n$$

## Rodrigues' rotation formula

In practice, we will not rely on the infinite series to compute the exponential:

$$\exp(t\mathbf{\Omega}) = e^{t\mathbf{\Omega}} = \sum_{n=0}^{\infty} \frac{1}{n!} (t\mathbf{\Omega})^n$$

The key property to observe is that skew-symmetric matrices are idempotent.

Given that  $\mathbf{\Omega}\mathbf{u} = \hat{\boldsymbol{\omega}}\mathbf{u} = \boldsymbol{\omega} \times \mathbf{u}$  for any vector  $\mathbf{u}$ , assuming that  $\boldsymbol{\omega}$  is unitary, we have:

$$\hat{\boldsymbol{\omega}}^2 = \boldsymbol{\omega}\boldsymbol{\omega}^T - \mathbf{Id}_3 \quad \hat{\boldsymbol{\omega}}^3 = -\hat{\boldsymbol{\omega}}$$

We can regroup the terms and use the infinite series of  $\sin(t)$  and  $1 - \cos(t)$ :

### Rodrigues' rotation formula

$$\exp(t\mathbf{\Omega}) = \mathbf{Id}_3 + \sin(t)\hat{\boldsymbol{\omega}} + [1 - \cos(t)]\hat{\boldsymbol{\omega}}^2$$

## Reciprocal application: the logarithm map

### Definition

The logarithm map of  $SO(3)$  maps a rotation matrix  $\mathbf{R}$  to a corresponding angular displacement  $t\mathbf{v}$ , with  $t \in \mathbb{R}^+$  and  $\|\mathbf{v}\| = 1$ , such that  $\mathbf{R} = \exp_{SO(3)}(t\mathbf{v})$ :

$$\log_{SO(3)} : SO(3) \longrightarrow \mathfrak{so}(3)$$

$$\mathbf{R} \longmapsto t\mathbf{v}$$

In practice, the logarithm can be computed as:

$$t = \|t\mathbf{v}\| = \cos^{-1} \left( \frac{\text{trace}(\mathbf{R}) - 1}{2} \right) \quad \mathbf{v} = \frac{t}{2 \sin(t)} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

It gives us a way to calculate a **distance between two rotations**:  $\text{dist}(\mathbf{R}_1, \mathbf{R}_2) = \|\log(\mathbf{R}_1^T \mathbf{R}_2)\|_2$ .

This will be useful when we seek to minimize the distance between two orientations.

# Outline

1.  $SO(2)$ : Rotations in 2D space
2.  $SE(2)$ : Rigid transformations in 2D space
3.  $SO(3)$ : Rotations in 3D space
4.  $SE(3)$ : Rigid transformations in 3D space
5. ★ Error log-linearity property

## The group $SE(3)$

The group of rigid transformations in dimension 3 is the *special Euclidean group* denoted  $SE(3)$ :

- **vector product preservation**  $v \times w$
- **preservation of the scalar product and therefore distances**

$$SE(3) = \{g : \mathbb{E}^3 \rightarrow \mathbb{E}^3, \forall x, y \in \mathbb{E}^3, \|g(x) - g(y)\| = \|x - y\|\}$$

The **rotation group  $SO(3)$**  is a subgroup of  **$SE(3)$** .

Thus, we denote:

$$SE(3) = \{(\mathbf{t}, \mathbf{R}), \mathbf{t} \in \mathbb{R}^3, \mathbf{R} \in SO(3)\}$$

It is not strictly the same as  $\mathbb{R}^3 \times SO(3)$ : we will see that there is a *coupling* between translation and rotation.

## Homogeneous coordinates

A rigid transform  $\mathbf{M} \in SE(3)$  acts on points  $\mathbf{p} \in \mathbb{R}^3$ :

$$\mathbf{M}(p) = \mathbf{R}(p) + \mathbf{t} \in \mathbb{E}^3$$

In the same way as  $SO(3)$  can be embedded into the group of orthonormal matrices,  $SE(3)$  can be embedded into the group of **invertible matrices**  $GL(4) \cong \mathbb{R}^{4 \times 4}$ :

$$\mathbf{M}(p) = \begin{bmatrix} R & t \\ \mathbf{0}_3 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

# Properties of homogeneous coordinates

Given the following homogeneous representation:

$$\mathbf{M}(p) = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}_3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix}$$

We have the following properties:

- **Composition:**  $\mathbf{M}_1 \circ \mathbf{M}_2 = \mathbf{M}_1 \mathbf{M}_2 = \begin{bmatrix} \mathbf{R}_1 & \mathbf{t}_1 \\ \mathbf{0}_3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_2 & \mathbf{t}_2 \\ \mathbf{0}_3 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1 \mathbf{R}_2 & \mathbf{t}_1 + \mathbf{R}_1 \mathbf{t}_2 \\ \mathbf{0}_3 & 1 \end{bmatrix}$
- **Identity:** The identity of SE(3) corresponds to  $\mathbf{I}_4$
- **Inverse:**  $\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T \mathbf{t} \\ \mathbf{0}_3 & 1 \end{bmatrix}$

This representation is the direct analog of the matrix representation  $\chi$  of SE(2).

## Linear and angular velocity space

In the same way as for rotations, an exponential map can be derived.

To do this, we're going to look at a curve  $\mathbf{M}(s) \in SE(3)$ :

$$\dot{\mathbf{M}}(s)\mathbf{M}(s)^{-1} = \begin{bmatrix} \dot{\mathbf{R}}(s)\mathbf{R}(s)^T & \dot{\mathbf{t}}(s) - \dot{\mathbf{R}}(s)\mathbf{R}^T(s)\mathbf{t}(s) \\ \mathbf{0}_3 & 1 \end{bmatrix}$$

We can recognize:

- the angular velocity  $\boldsymbol{\Omega}(s) = \dot{\mathbf{R}}(s)\mathbf{R}^T(s) = \boldsymbol{\omega}(s) \times$
- the linear velocity  $\mathbf{v}(s) = \dot{\mathbf{t}}(s) - \boldsymbol{\omega}(s) \times \mathbf{t}(s)$

Thus, the tuple  $(\mathbf{v}, \boldsymbol{\omega}) \in \mathfrak{se}(3) \cong \mathbb{R}^3 \times \mathfrak{so}(3)$  is the tangent vector to  $\mathbf{M} \in SE(3)$ .

## Exponential map

From the relation that links  $\mathbf{M}(s)$  to its time derivative  $\dot{\mathbf{M}}(s)$ :

$$\dot{\mathbf{M}}(s)\mathbf{M}(s)^{-1} = \begin{bmatrix} \dot{\mathbf{R}}(s)\mathbf{R}^T(s) & \dot{t}(s) - \dot{\mathbf{R}}(s)\mathbf{R}^T(s)t(s) \\ \mathbf{0}_3 & 1 \end{bmatrix}$$

we can follow derived the exponential map:

$$\exp_{SE(3)}(v, w) = \begin{bmatrix} \exp_{\mathfrak{so}(3)}(\boldsymbol{\omega}) & \frac{1}{\|\boldsymbol{\omega}\|} (\boldsymbol{\omega}\boldsymbol{\omega}^T \mathbf{v} + (\mathbf{Id}_3 - \exp_{\mathfrak{so}(3)}(\boldsymbol{\omega}))\boldsymbol{\omega} \times \mathbf{v}) \\ \mathbf{0}_3 & 1 \end{bmatrix}$$

$$\exp_{SE(3)} : \frac{\mathfrak{se}(3) \longrightarrow SE(3)}{(v, w) \longmapsto M}$$

## Reciprocal application: the logarithm map

The exponential map  $\exp_{SE(3)}$  is surjective. We can construct its reciprocal:

$$\log_{SE(3)} : \frac{SE(3) \longrightarrow \mathfrak{se}(3)}{M \longmapsto (\mathbf{v}, \boldsymbol{\omega})}$$

with:

$$\boldsymbol{\omega} = \log_{SO(3)}(\mathbf{R}) \quad \mathbf{v} = \|\boldsymbol{\omega}\|(\boldsymbol{\omega}\boldsymbol{\omega}^T + (\mathbf{Id}_3 - \mathbf{R})\boldsymbol{\Omega})^{-1}t$$

Here as well we can define a **distance** between two frames:

$$\text{dist}(\mathbf{M}_1, \mathbf{M}_2) = \|\log(\mathbf{M}_1^{-1}\mathbf{M}_2)\|_2$$

## Reference

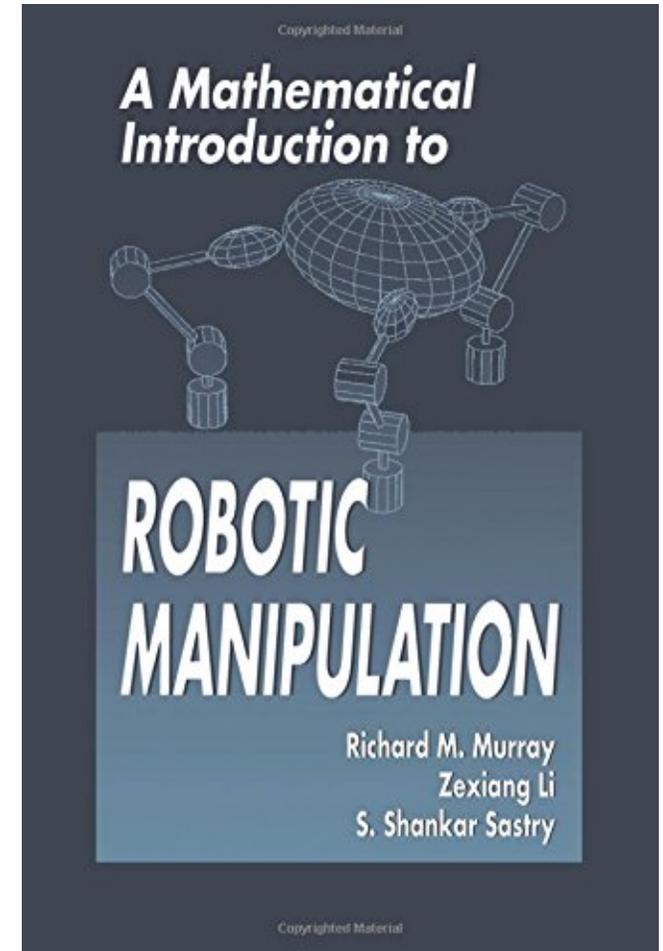
### **A Mathematical Introduction to Robotic Manipulation.**

Richard M. Murray, Zexiang Li, S. Shankar Sastry.

California Institute of Technology, Hong Kong University of Science and Technology, University of California, Berkeley.

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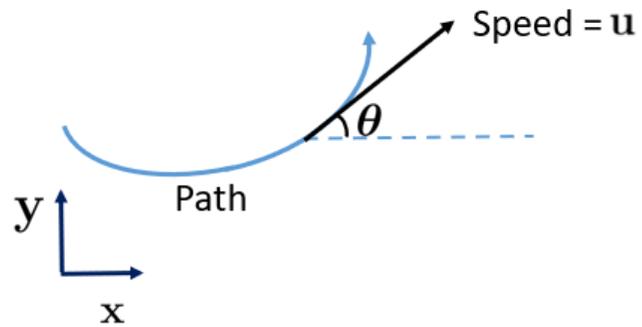
Available from: <http://www.cds.caltech.edu/~murray/mlswiki>



# Outline

1.  $SO(2)$ : Rotations in 2D space
2.  $SE(2)$ : Rigid transformations in 2D space
3.  $SO(3)$ : Rotations in 3D space
4.  $SE(3)$ : Rigid transformations in 3D space
5. ★ [Error log-linearity property](#)

# The unicycle equations



## Definition

The unicycle equations write

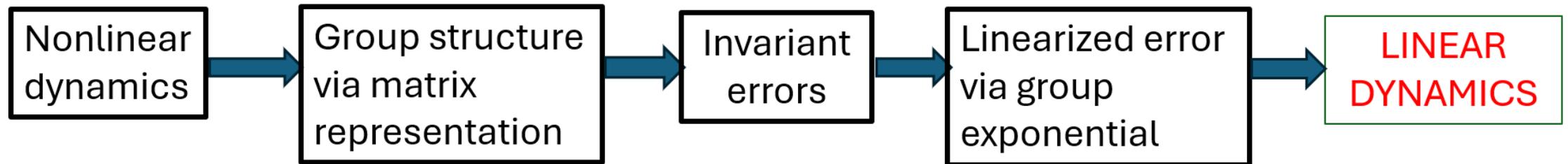
$$\frac{d}{dt}\theta = \omega$$

$$\frac{d}{dt}x = u \cos \theta$$

$$\frac{d}{dt}y = u \sin \theta$$

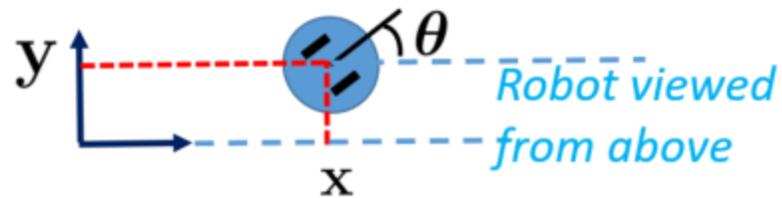
## What we are about to do

The "goal" of invariant filtering is to reveal a "hidden structure" that underlies estimation problems.



There is hidden linearity, but in terms of *error dynamics*, i.e., discrepancy between solutions.

## The unicycle equations



$$\frac{d}{dt}\theta = \omega \quad \frac{d}{dt}x = u \cos \theta \quad \frac{d}{dt}y = u \sin \theta$$

- They model the motion of a 2D robot whose wheels roll but do not slip.
- $u$  is the average of wheel speeds, and  $\omega$  is obtained as a difference.

Let  $p := \begin{bmatrix} x \\ y \end{bmatrix}$  and  $v := \begin{bmatrix} u \\ 0 \end{bmatrix} \in \mathbb{R}^2$ , as we have  $\dot{p} = R(\theta)v$ , the equations rewrite on the group  $SE(2)$

$$\frac{d}{dt}\chi_t = \chi_t \Omega_t, \quad \Omega := \begin{bmatrix} \omega \mathbf{J} & v \\ 0 & 0 \end{bmatrix}, \quad \chi_t := \begin{bmatrix} R(\theta_t) & p_t \\ 0 & 1 \end{bmatrix}$$

## The unicycle equations

They rewrite on the group  $SE(2)$

$$\frac{d}{dt}\chi_t = \chi_t\Omega_t$$

### What we have not

When  $\Omega_t \equiv \bar{\Omega}$  constant, we have  $\chi_t = \chi_0 \exp(\bar{\Omega}t)$  However, with  $\Omega_t$  nonconstant *nothing of that kind holds*.

### What we do have

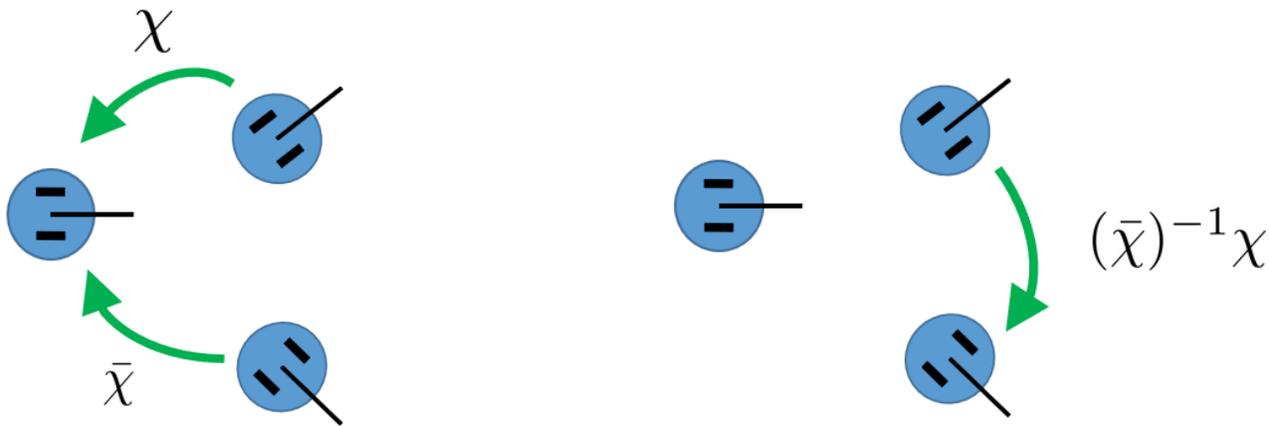
The theoretical properties come with the notion of invariant error.

# Invariant errors

## Definition

The left-invariant error on  $SE(2)$  between  $\chi$  and  $\bar{\chi}$  is the element represented by the matrix

$$E := (\bar{\chi})^{-1}\chi$$



Invariance to left multiplications: a global change of frame rotates and translates everyone but leaves the relative error unchanged.

## Invariant errors

### Definition

The left-invariant error on  $SE(2)$  between  $\chi$  and  $\bar{\chi}$  is the element represented by the matrix

$$E := (\bar{\chi})^{-1}\chi$$

Similarly the right-invariant error writes

$$e := \chi(\bar{\chi})^{-1}$$

$E$  is invariant to left multiplication  $\bar{\chi} \mapsto G\bar{\chi}$  and  $\chi \mapsto G\chi$ . We have

$$E = \begin{bmatrix} R(-\bar{\theta}) & -R(-\bar{\theta})\bar{p} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(\theta) & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R(\tilde{\theta}) & R(-\bar{\theta})\tilde{p} \\ 0 & 1 \end{bmatrix}$$

with  $\tilde{\theta} := \theta - \bar{\theta}$  and  $\tilde{p} := p - \bar{p}$ . We let  $E_\theta := \tilde{\theta}$  and  $E_p := R(-\bar{\theta})\tilde{p}$ .

## The unicycle: error equation in the original variables

Consider 2 trajectories  $(\theta(t), p(t))_{t \geq 0}$  and  $(\bar{\theta}(t), \bar{p}(t))_{t \geq 0}$  of the unicycle

$$\frac{d}{dt}\theta = \omega(t), \quad \frac{d}{dt}p = R(\theta)v(t), \quad \text{and} \quad \frac{d}{dt}\bar{\theta} = \omega(t), \quad \frac{d}{dt}\bar{p} = R(\bar{\theta})v(t)$$

Consider the error (discrepancy) in the original variables  $\tilde{\theta} := \theta - \bar{\theta}$  and  $\tilde{p} := p - \bar{p}$ . The error equation writes

$$\frac{d}{dt}\tilde{\theta} = \omega(t) - \omega(t) = 0, \quad \frac{d}{dt}\tilde{p} = [R(\theta) - R(\bar{\theta})]v(t)$$

Contrary to the linear case, it is **not** a function of the error  $(\tilde{\theta}, \tilde{p})$  and explicitly depends on  $\theta(t)$  as

$$R(\theta) - R(\bar{\theta}) \neq R(\theta - \bar{\theta}) = R(\tilde{\theta})$$

## The unicycle: error equation in the invariant error variables

Consider 2 trajectories  $(\theta(t), p(t))_{t \geq 0}$  and  $(\bar{\theta}(t), \bar{p}(t))_{t \geq 0}$  of the unicycle

$$\frac{d}{dt}\theta = \omega, \quad \frac{d}{dt}p = R(\theta)v, \quad \text{and} \quad \frac{d}{dt}\bar{\theta} = \omega, \quad \frac{d}{dt}\bar{p} = R(\bar{\theta})v$$

Recall the definition of the left-invariant error

$$E = \begin{bmatrix} R(-\bar{\theta}) & -R(-\bar{\theta})\bar{p} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(\theta) & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R(\tilde{\theta}) & R(-\bar{\theta})\tilde{p} \\ 0 & 1 \end{bmatrix}$$

We may accordingly define  $E_\theta := \tilde{\theta} = \theta - \bar{\theta}$  and  $E_p := R(-\bar{\theta})\tilde{p}$ .

## The unicycle: error equation in the invariant error variables

Consider 2 trajectories  $(\theta(t), p(t))_{t \geq 0}$  and  $(\bar{\theta}(t), \bar{p}(t))_{t \geq 0}$  of the unicycle

$$\frac{d}{dt}\theta = \omega, \quad \frac{d}{dt}p = R(\theta)v, \quad \text{and} \quad \frac{d}{dt}\bar{\theta} = \omega, \quad \frac{d}{dt}\bar{p} = R(\bar{\theta})v$$

Consider the (left) invariant error (discrepancy)  $E_\theta := \tilde{\theta} = \theta - \bar{\theta}$  and  $E_p := R(-\bar{\theta})\tilde{p}$ . The error equation now writes

$$\begin{aligned} \frac{d}{dt}E_\theta &= 0 \\ \frac{d}{dt}E_p &= \left(\frac{d}{dt}R(-\theta)\right)\tilde{p} + R(-\bar{\theta})\frac{d}{dt}\tilde{p} = -\omega\mathbf{J}R(-\bar{\theta})\tilde{p} + R(-\bar{\theta})[R(\theta) - R(\bar{\theta})]v \\ &= -\omega\mathbf{J}E_p + [R(\tilde{\theta}) - Id]v = -\omega\mathbf{J}E_p + [R(E_\theta) - Id]v \end{aligned}$$

Akin to the linear case, it **is** a function of the error  $(E_\theta, E_p)$  and does not explicitly depends on  $\theta(t), p(t)$ .

## Highlight of the show: the log-linear error property

The invariant error equation writes

$$\frac{d}{dt}E_\theta = 0, \quad \frac{d}{dt}E_p = -\omega\mathbf{J}E_p + [R(E_\theta) - Id]v$$

### Definition

The "linearized" error  $(\xi_\theta, \xi_p) \in \mathbb{R} \times \mathbb{R}^2$  is defined as the  $SE(2)$ -log of the error

$$\exp_{SE(2)}(\xi_\theta, \xi_p) = E = \begin{bmatrix} R(E_\theta) & E_p \\ 0 & 1 \end{bmatrix}$$

This means by definition

$$E_\theta = \tilde{\theta} = \xi_\theta, \quad E_p = \mathbf{V}(\xi_\theta)\xi_p$$

where we recall that  $Id + \alpha\mathbf{J}\mathbf{V}(\alpha) = R(\alpha)$ .

## Highlight of the show: the log-linear error property

The invariant error equation writes

$$\frac{d}{dt} E_\theta = 0, \quad \frac{d}{dt} E_p = -\omega \mathbf{J} E_p + [R(E_\theta) - Id]v \quad (*)$$

The linearized error is a vector, defined by

$$\xi_\theta = E_\theta, \quad E_p = \mathbf{V}(\xi_\theta) \xi_p \quad (\#)$$

where we recall  $R(\xi_\theta) - Id = \xi_\theta \mathbf{J} \mathbf{V}(\xi_\theta)$ . Let us use it the relation in (\*).

$$\frac{d}{dt} \xi_\theta = 0, \quad \frac{d}{dt} E_p = -\omega \mathbf{J} E_p + \xi_\theta \mathbf{J} \mathbf{V}(\xi_\theta) v$$

Finally replacing  $E_p$  using (#) and as  $\frac{d}{dt} \mathbf{V}(\xi_\theta) = 0$ , we have

$$\frac{d}{dt} [\mathbf{V}(\xi_\theta) \xi_p] = -\omega \mathbf{J} \mathbf{V}(\xi_\theta) \xi_p + \xi_\theta \mathbf{J} \mathbf{V}(\xi_\theta) v$$

Factorizing by  $\mathbf{V}(\xi_\theta)$  and using that  $\mathbf{J}$  and  $\mathbf{V}$  commute (as  $\mathbf{V}(\omega) = \mathbf{I}_2 + \frac{1}{2!} (\omega \mathbf{J}) + \frac{1}{3!} (\omega \mathbf{J})^2 + \dots$ ) yields  $\frac{d}{dt} \xi_p = -\omega \mathbf{J} \xi_p + \mathbf{J} v \xi_\theta$ .

## Highlight of the show: the log-linear error property

We have proved the following result.

### Theorem (Error log-linearity property)

Consider the (nonlinear) error equation

$$\frac{d}{dt}E_\theta = 0, \quad \frac{d}{dt}E_p = -\omega\mathbf{J}E_p + [R(E_\theta) - Id]v$$

Define the linearized error by

$$\xi_\theta = E_\theta, \quad E_p = \mathbf{V}(\xi_\theta)\xi_p$$

The ODE above is **exactly** equivalent to the **linear** ODE

$$\frac{d}{dt} \begin{bmatrix} \xi_\theta \\ \xi_p \end{bmatrix} = \begin{bmatrix} 0 & 0_{1 \times 2} \\ \mathbf{J}v & -\omega\mathbf{J} \end{bmatrix} \begin{bmatrix} \xi_\theta \\ \xi_p \end{bmatrix}$$

## Consequence: the banana distribution explained

- We have in particular

$$(\xi_\theta(0), \xi_p(0)) \sim \mathcal{N}(0, \mathbf{P}_0) \Rightarrow (\xi_\theta(t), \xi_p(t)) \sim \mathcal{N}(0, \mathbf{P}_t)$$

as linear equations preserve Gaussians.

- An initial uncertainty in the form

$$\chi_0 = \mu_0 \exp_{SE(2)}(\xi_\theta(0), \xi_p(0)), \quad (\xi_\theta(0), \xi_p(0)) \sim \mathcal{N}(0, \mathbf{P}_0)$$

yields

$$\chi_t = \mu_t \exp_{SE(2)}(\xi_\theta(t), \xi_p(t)), \quad (\xi_\theta(t), \xi_p(t)) \sim \mathcal{N}(0, \mathbf{P}_t)$$

where  $\mu(t)$  is a solution of the system's equations initialized at  $\mu_0$ .

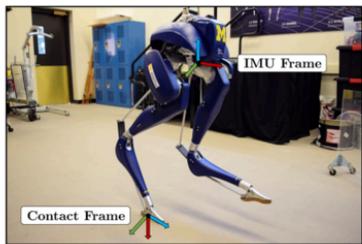
**Proof:**

$$(\mu_t)^{-1} \chi_t = \exp_{SE(2)}(\xi_\theta(t), \xi_p(t))$$

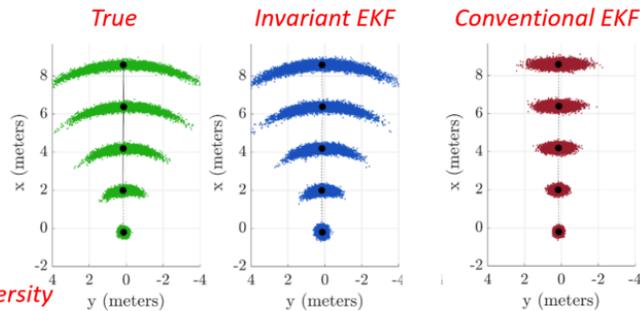
is a left-invariant error indeed. It verifies the log-linear property.

# Consequence: the banana distribution explained

- This explains the banana distribution is wholly captured: the banana distribution is Gaussian in exponential coordinates<sup>1</sup>.
- The theory of invariant filtering much generalizes the result.



Cassie robot, Robotics Institute, University of Michigan (courtesy of Jessy Grizzle)

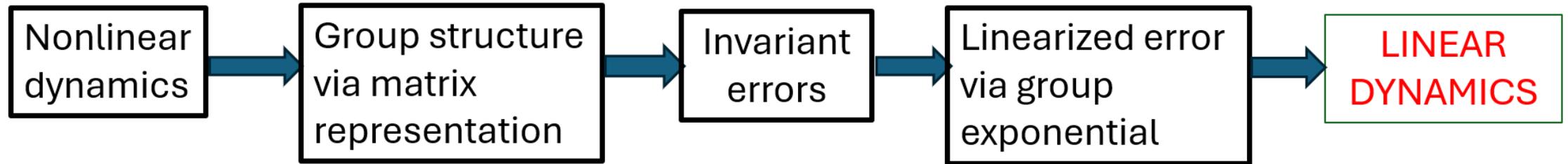


Ross Hartley, Maani Ghaffari, Ryan M Eustice & Jessy W Grizzle. Contact-aided invariant extended Kalman filtering for robot state estimation. *The International Journal of Robotics Research*. 2020.

<sup>1</sup> Long, A. W., Wolfe, K. C., Mashner, M. J., & Chirikjian, G. S. (2013). The banana distribution is Gaussian: A localization study with exponential coordinates.

## Summary

For the unicycle equations we followed the scheme



Consequence: Methods based on linearizations (such as the EKF) do not incur any error.

This explains the banana-shaped dispersion under the dynamics, and besides allows for a convergence proof of the IEKF.

This scheme extends across the entire invariant filtering theory.

## Final slide: Where does it stem from? How general??

In invariant filtering theory, we target nonlinear dynamics that have an automorphism property. Everything stems from the classical result:

### Theorem (Lie group - Lie algebra correspondence)

Suppose that  $\phi : G \rightarrow G$  is a Lie group automorphism. Then there exists a unique **linear** map  $\tilde{\phi} : \mathfrak{g} \rightarrow \mathfrak{g}$  such that

$$\phi(\exp_G(\xi)) = \exp_G(\tilde{\phi}(\xi))$$

We have hence turned into "automorphism hunters". See Barrau and Bonnabel "Linear observed systems on groups", SCL, 2019, for more info.

Notably we introduced so-called "group affine dynamics" and proved:

$$\text{Group affine dynamics} \Leftrightarrow \text{Error - automorphism} \Rightarrow \text{log-linearity}$$