Divergent Components of Motion

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October 29, 2019

JRL Seminar, CNRS-AIST Joint Robotics Laboratory, Tsukuba

WARMUP : SOME CONTROL THEORY

$$\dot{x} = \lambda x$$

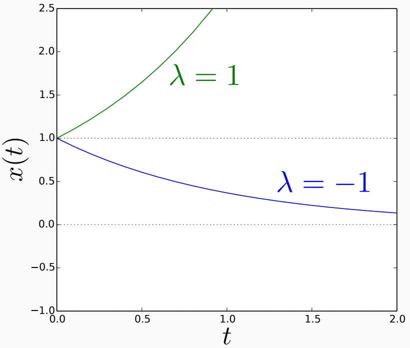
Autonomous system

$$\dot{x} = \lambda x$$

Trajectories

$$x(t) = e^{\lambda t} x(0)$$

- We want to control $x \to 0$
- Pole of the system : λ
- Stability : $x \underset{t \to \infty}{\longrightarrow} 0$ iff $\lambda < 0$



To control x toward a time-varying reference $x^d(t)$, apply the same to the error $e = x - x^d$

$$\dot{x} = Ax$$

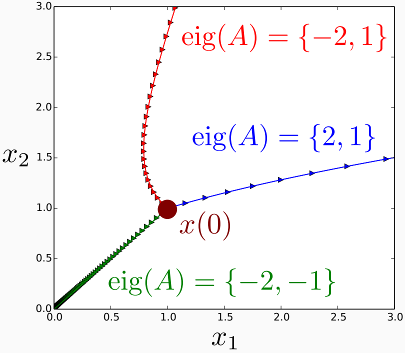
Autonomous system

$$\dot{x} = Ax$$

Trajectories

$$x(t) = e^{At}x(0)$$

- Poles : $\mathrm{eig}(A) = \{\lambda_1, \ldots, \lambda_n\}$
- Stability : all poles $\lambda_i < 0$



$$\dot{x} = Ax + Bu$$

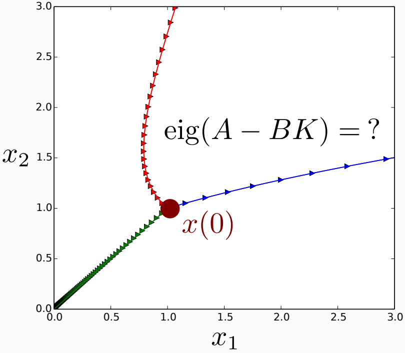
Linear system

$$\dot{x} = Ax + Bu$$

Linear feedback

$$u = -Kx$$

- Closed loop : $\dot{x} = (A - BK)x$
- Poles : $\mathrm{eig}(A - BK)$
- Pole placement



APPLICATION TO LEGGED ROBOTS

Equation of motion

$$\ddot{c} = \omega^2(c - z)$$

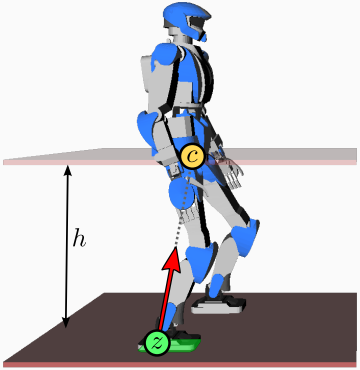
Real robot and reference trajectories :

$$\ddot{c} = \omega^2(c - z)$$
$$\ddot{c}^d = \omega^2(c^d - z^d)$$

Tracking error $\Delta y = y - y^d$:

$$\Delta\ddot{c} = \omega^2(\Delta c - \Delta z)$$

Linear system : error dynamics is the same as system dynamics.



1. Shuuji Kajita, Fumio Kanehiro, Kenji Kaneko, Kazuhito Yokoi et Hirohisa Hirukawa. « The 3D Linear Inverted Pendulum Mode : A simple modeling for a biped walking pattern generation ». In : *IEEE/RSJ International Conference on Intelligent Robots and Systems*. 2001.

Equation of motion

$$\ddot{c} = \omega^2(c - z)$$

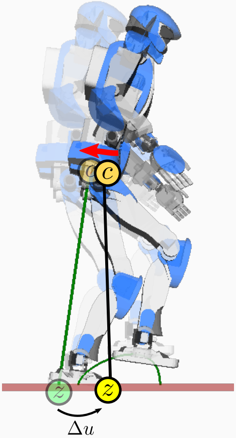
With $x = [c\ \dot{c}]$ and $u = z$:

$$\dot{x} = \begin{bmatrix} \dot{c} \\ \ddot{c} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \omega^2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -\omega^2 \end{bmatrix} u$$

Linear feedback :

$$\Delta u = k_p \Delta c + k_v \Delta \dot{c}$$

How to choose the gains k_p and k_v ?



2. Shuuji KAJITA, Mitsuharu MORISAWA, Kanako MIURA, Shin'ichiro NAKAOKA, Kensuke HARADA, Kenji KANEKO, Fumio KANEHIRO et Kazuhito YOKOI. « Biped walking stabilization based on linear inverted pendulum tracking ». In : *IEEE/RSJ International Conference on Intelligent Robots and Systems*. 2010.

What if we can diagonalize A ?

Linear system

$$\dot{x} = Ax + Bu \qquad u = -Kx$$

The diagonalization of A would give :

$$A = \begin{bmatrix} 0 & 1 \\ \omega^2 & 0 \end{bmatrix} = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}$$

We could then change variable $\widetilde{x} = P^{-1}x$ so that :

$$\dot{\widetilde{x}} = P^{-1}\dot{x} = P^{-1}AP\widetilde{x} + P^{-1}Bu = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \widetilde{x} + \widetilde{B}\widetilde{u}$$

The coordinates of $\widetilde{x}$ are *decoupled*.

Linear system

$$\dot{x} = Ax + Bu \qquad u = -Kx$$

Good news ! We can diagonalize A :

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -\omega & \omega \end{bmatrix}\begin{bmatrix} -\omega & 0 \\ 0 & +\omega \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{\omega} \\ 1 & +\frac{1}{\omega} \end{bmatrix}$$

We then change variable :

$$\widetilde{x} = \begin{bmatrix} \zeta \\ \xi \end{bmatrix} = P^{-1}x = \begin{bmatrix} c - \frac{\dot{c}}{\omega} \\ c + \frac{\dot{c}}{\omega} \end{bmatrix}$$

So that our system becomes :

$$\dot{\widetilde{x}} = \begin{bmatrix} \dot{\zeta} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} -\omega & 0 \\ 0 & +\omega \end{bmatrix}\begin{bmatrix} \zeta \\ \xi \end{bmatrix} + \begin{bmatrix} +\omega \\ -\omega \end{bmatrix} u \Longleftrightarrow \begin{cases} \dot{\zeta} = \omega(u - \zeta) \\ \dot{\xi} = \omega(\xi - u) \end{cases}$$

Convergent component of motion

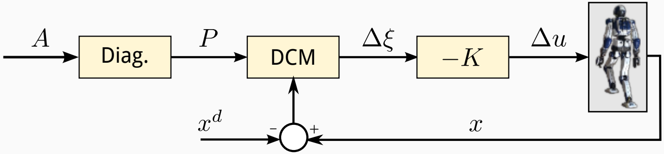
$$\dot{\zeta} = \omega(u - \zeta)$$

- No feedback required
- Stable : $\zeta \to 0$ as long as $u \to 0$

Divergent component of motion

$$\dot{\xi} = \omega(\xi - u)$$

- Feedback $u = k_\xi \xi$ required !
- Stable as long as $k_\xi > 1$



We only apply $u = k_\xi \xi$ since $\xi \to 0 \Rightarrow u \to 0 \Rightarrow \zeta \to 0$.

Equation of motion

$$\ddot{c} = \omega^2 (c - z)$$

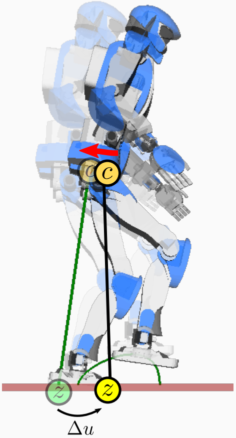
With $x = [c\ \dot{c}]$ and $u = z$:

$$\dot{x} = \begin{bmatrix} \dot{c} \\ \ddot{c} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \omega^2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -\omega^2 \end{bmatrix} u$$

We now know how to select our gains :

$$u = k_p c + k_v \dot{c} = k_\xi \xi$$
$$k_v = k_p / \omega$$
$$k_p > 1$$

Best possible linear feedback [Sug09].



3. Tomomichi Sugihara. « Standing stabilizability and stepping maneuver in planar bipedalism based on the best COM-ZMP regulator ». In : *IEEE International Conference on Robotics and Automation*. 2009.

$$\dot{x} = A(t)x$$

The idea behind this approach has been explored in control theory.

Exponential dichotomy (Coppel, 1967)

The system $\dot{x} = A(t)x$ has an *exponential dichotomy* iff there exists a projection Π and constants $K, L, \alpha, \beta > 0$ such that its solutions satisfy :

$$
\begin{aligned}
|\Pi x(t)| &\leq K e^{-\alpha(t-t_0)} |\Pi x(t_0)| & \qquad t_0 \leq t \\
|(I - \Pi)x(t)| &\geq L e^{+\beta(t-t_0)} |(I - \Pi)x(t_0)| & \qquad t_0 \leq t
\end{aligned}
$$

- Π projects on CCMs that converge in forward time
- $(I - \Pi)$ projects on DCMs that diverge in forward time

Pendular models for locomotion fall into this category of systems.

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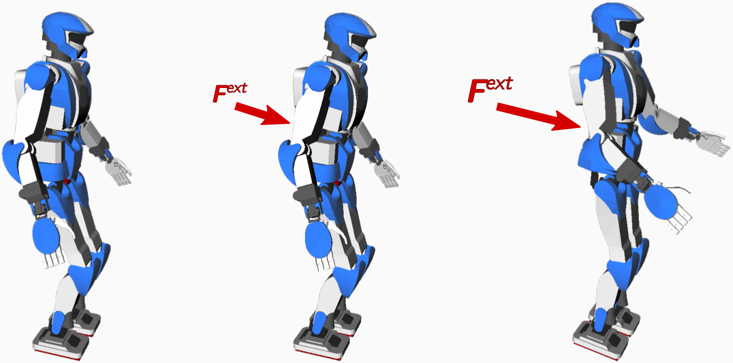
4. William A. Coppel. *Dichotomies in stability theory*. Lecture notes in mathematics 629. OCLC : 3609707. Berlin : Springer, 1978. 97 p.

$$\dot{x} = f(x)$$

HEIGHT VARIATION STRATEGY



Rest position Ankle strategy Height variation strategy

5. Twan KOOLEN, Michael POSA et Russ TEDRAKE. « Balance control using center of mass height variation : Limitations imposed by unilateral contact ». In : *IEEE-RAS International Conference on Humanoid Robots*. 2016.

Equation of motion

$$\ddot{c} = \lambda(c - z) + g$$

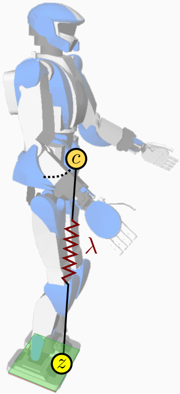
Nonlinear : both z and λ are inputs

$$\dot{x} = \begin{bmatrix} \dot{c} \\ \ddot{c} \end{bmatrix} = \begin{bmatrix} 0 & I_3 \\ \lambda I_3 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -I_3 \end{bmatrix} (\lambda z - g)$$
$$= A(\lambda) x + B(\lambda z - g)$$

Planning : fix $\lambda(t)$ to make system linear time-variant :

$$\dot{x} = A(t)x + B\lambda(t)z - Bg$$

What about feedback on λ ?



6. Michael A. Hopkins, Dennis W. Hong et Alexander Leonessa. « Humanoid locomotion on uneven terrain using the time-varying Divergent Component of Motion ». In : *IEEE-RAS International Conference on Humanoid Robots*. IEEE, 2014, p. 266–272.

Equation of motion

$$\dot{x} = \begin{bmatrix} \dot{c} \\ \ddot{c} \end{bmatrix} = \begin{bmatrix} 0 & I_3 \\ \lambda I_3 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -I_3 \end{bmatrix} (\lambda z - g)$$

Let $\widetilde{x} = P^{-1}x$ with :

$$P = \frac{1}{\gamma + \omega} \begin{bmatrix} I_3 & I_3 \\ -\omega I_3 & +\gamma I_3 \end{bmatrix} \Longleftrightarrow P^{-1} = \begin{bmatrix} \gamma I_3 & -I_3 \\ \omega I_3 & +I_3 \end{bmatrix}$$

Changing variable yields $\dot{\widetilde{x}} = \widetilde{A}x + \widetilde{b}$ with :

$$\widetilde{A} = \frac{1}{\gamma + \omega} \begin{bmatrix} (\dot{\gamma} - \gamma\omega - \lambda)I_3 & (\dot{\gamma} + \gamma^2 - \lambda)I_3 \\ (\dot{\omega} - \omega^2 + \lambda)I_3 & (\dot{\omega} + \omega\gamma + \lambda)I_3 \end{bmatrix}$$

We have a dichotomy if $\widetilde{A}$ is diagonal...

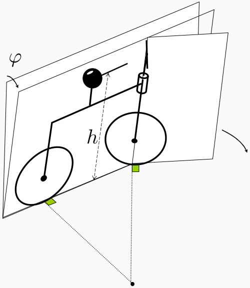


FIGURE 1 : Figure adapted from [HSF04].

7. J. HAUSER, A. SACCON et R. FREZZA. « Achievable motorcycle trajectories ». In : *IEEE Conference on Decision and Control*. Nassau, Bahamas : IEEE, 2004, 3944–3949 Vol.4.

Equation of motion

$$\dot{x} = \begin{bmatrix} \dot{c} \\ \ddot{c} \end{bmatrix} = \begin{bmatrix} 0 & I_3 \\ \lambda I_3 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -I_3 \end{bmatrix} (\lambda z - g)$$

So let's make $\widetilde{A}$ diagonal ! Choose :

$$\dot{\gamma} = \lambda - \gamma^2$$
$$\dot{\omega} = \omega^2 - \lambda$$

Then we have an exponential dichotomy :

$$\zeta = c - \frac{\dot{c}}{\gamma} \qquad \dot{\zeta} = \frac{\lambda}{\gamma}(z - \zeta) - \frac{g}{\gamma}$$
$$\xi = c + \frac{\dot{c}}{\omega} \qquad \dot{\xi} = \frac{\lambda}{\omega}(\xi - z) + \frac{g}{\omega}$$

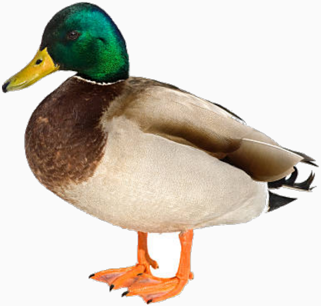
We achieved the exponential dichotomy by choosing :

$$\dot{\omega} = \omega^2 - \lambda$$

This natural frequency ω behaves like a DCM :

- λ is an input
- ω diverges under constant input

8. Stéphane CARON, Adrien ESCANDE, Leonardo LANARI et Bastien MALLEIN. « Capturability-based Pattern Generation for Walking with Variable Height ». In : *IEEE Transactions on Robotics* (juil. 2019).



https://en.wikipedia.org/wiki/Duck_test

Divergent component of motion

$$x = \begin{bmatrix} \xi \\ \omega \end{bmatrix} \implies \dot{x} = \frac{1}{\omega} \begin{bmatrix} \lambda I_3 & 0 \\ 0 & \omega^2 \end{bmatrix} x - \frac{1}{\omega} \begin{bmatrix} \lambda I_3 & 0 \\ 0 & \omega \end{bmatrix} \begin{bmatrix} z \\ \lambda \end{bmatrix} + \frac{1}{\omega} \begin{bmatrix} g \\ 0 \end{bmatrix}$$

Nonlinear system.

9. Stéphane CARON. « Biped Stabilization by Linear Feedback of the Variable-Height Inverted Pendulum Model ». submitted. Sept. 2019.

Divergent component of motion

$$x = \begin{bmatrix} \xi \\ \omega \end{bmatrix} \implies \dot{x} = \frac{1}{\omega} \begin{bmatrix} \lambda I_3 & 0 \\ 0 & \omega^2 \end{bmatrix} x - \frac{1}{\omega} \begin{bmatrix} \lambda I_3 & 0 \\ 0 & \omega \end{bmatrix} \begin{bmatrix} z \\ \lambda \end{bmatrix} + \frac{1}{\omega} \begin{bmatrix} g \\ 0 \end{bmatrix}$$

Nonlinear system.

Take its linearized error dynamics :

$$\Delta \dot{x} = \frac{1}{\omega^d} \begin{bmatrix} \lambda^d I_3 & -\ddot{c}^d/\omega^d \\ 0 & 2(\omega^d)^2 \end{bmatrix} \Delta x - \frac{1}{\omega^d} \begin{bmatrix} \lambda^d I_3 & (\xi^d - z^d) \\ 0 & \omega^d \end{bmatrix} \begin{bmatrix} \Delta z \\ \Delta \lambda \end{bmatrix}$$

We are back to $\Delta \dot{x} = A \Delta x + B \Delta u$.[9]

9. Stéphane Caron. « Biped Stabilization by Linear Feedback of the Variable-Height Inverted Pendulum Model ». submitted. Sept. 2019.

Place *poles* of the closed-loop system :

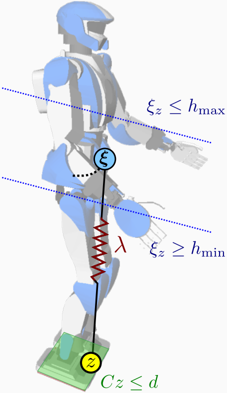
$$\Delta\dot{x}^* = (1-k)\frac{1}{\omega^d}\begin{bmatrix}\lambda^d I_3 & 0 \\ 0 & (\omega^d)^2\end{bmatrix}\Delta x$$

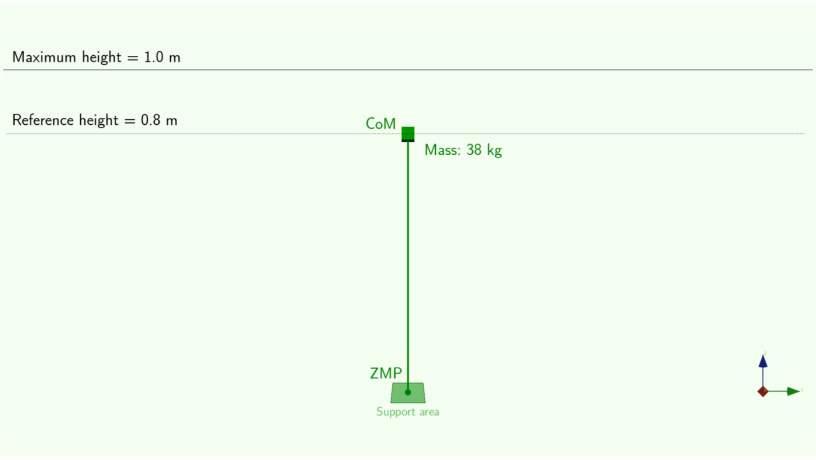
Minimize : $\|\Delta\dot{x} - \Delta\dot{x}^*\|^2$
Subject to :

- Linearized dynamics : $\Delta\dot{x} = A\Delta x + B\Delta u$
- ZMP support area : $C(z^d + \Delta z) \leq d$
- Reaction force : $\lambda_{min} \leq \lambda^d + \Delta\lambda \leq \lambda_{max}$
- Kinematics : $h_{min} \leq \xi_z^d + \Delta\xi_z \leq h_{max}$

Least squares problem :

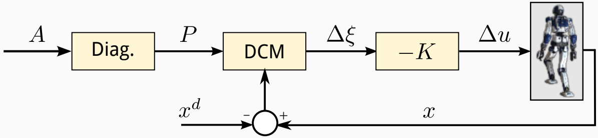
$\Rightarrow$ Constrained optimal gains $\Delta u = -K\Delta x$



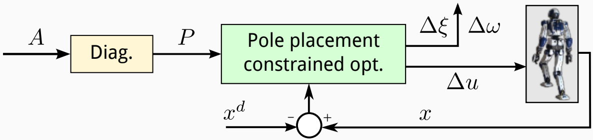


https://github.com/stephane-caron/pymanoid/blob/master/examples/vhip_stabilization.py

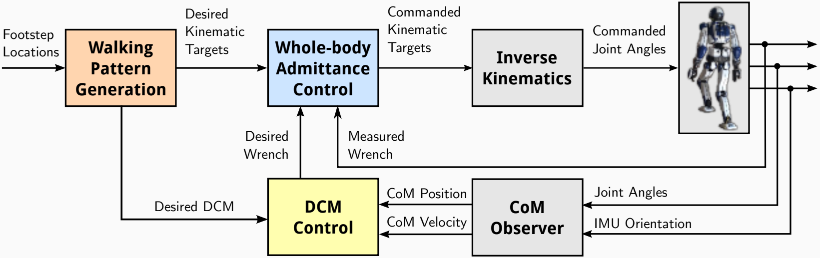
One last observation : previously, the DCM was a measure :



Now, the controller **adjusts** the DCM based on state and constraints :



tests on HRP-4



Footstep
Locations
Walking
Pattern
Generation
Desired
Kinematic
Targets
Whole-body
Admittance
Control
Commanded
Kinematic
Targets
Inverse
Kinematics
Commanded
Joint Angles
Desired
Wrench
Measured
Wrench
Desired DCM
DCM
Control
CoM Position
CoM Velocity
CoM
Observer
Joint Angles
IMU Orientation

LIP tracking

VHIP tracking

https://github.com/stephane-caron/vhip_walking_controller

BALANCE CONTROL

https://github.com/stephane-caron/vhip_walking_controller

WRAP-UP

- Ground zero : $\dot{x} = \lambda x$
- Linear time invariant : $\dot{x} = Ax$
- Linear feedback : $\dot{x} = Ax + Bu$
- Diagonalize A : apply feedback to divergent components
- General case : *exponential dichotomy*
- If it looks like a DCM, it is a DCM !
- Linearized error dynamics
- Least-squares pole placement
- Ask the robot !

Thank you for your attention!

Thanks to Dr Kajita and Mr Vasalya for correcting my mistakes in previous versions of these slides, and to Dr Escande for suggesting improvements.

Discussion : `https://scaron.info/talks/jrl-2019.html#discussion`

[Car+19] Stéphane CARON, Adrien ESCANDE, Leonardo LANARI et Bastien MALLEIN. « Capturability-based Pattern Generation for Walking with Variable Height ». In : *IEEE Transactions on Robotics* (juil. 2019).

[Car19] Stéphane CARON. « Biped Stabilization by Linear Feedback of the Variable-Height Inverted Pendulum Model ». submitted. Sept. 2019.

[Cop78] William A. COPPEL. *Dichotomies in stability theory*. Lecture notes in mathematics 629. OCLC : 3609707. Berlin : Springer, 1978. 97 p.

[HHL14] Michael A. HOPKINS, Dennis W. HONG et Alexander LEONESSA. « Humanoid locomotion on uneven terrain using the time-varying Divergent Component of Motion ». In : *IEEE-RAS International Conference on Humanoid Robots*. IEEE, 2014, p. 266–272.

[HSF04] J. HAUSER, A. SACCON et R. FREZZA. « Achievable motorcycle trajectories ». In : *IEEE Conference on Decision and Control*. Nassau, Bahamas : IEEE, 2004, 3944–3949 Vol.4.

[Kaj+01] Shuuji KAJITA, Fumio KANEHIRO, Kenji KANEKO, Kazuhito YOKOI et Hirohisa HIRUKAWA. « The 3D Linear Inverted Pendulum Mode : A simple modeling for a biped walking pattern generation ». In : *IEEE/RSJ International Conference on Intelligent Robots and Systems*. 2001.

[Kaj+10] Shuuji KAJITA, Mitsuharu MORISAWA, Kanako MIURA, Shin'ichiro NAKAOKA, Kensuke HARADA, Kenji KANEKO, Fumio KANEHIRO et Kazuhito YOKOI. « Biped walking stabilization based on linear inverted pendulum tracking ». In : *IEEE/RSJ International Conference on Intelligent Robots and Systems*. 2010.

[KPT16] Twan KOOLEN, Michael POSA et Russ TEDRAKE. « Balance control using center of mass height variation : Limitations imposed by unilateral contact ». In : *IEEE-RAS International Conference on Humanoid Robots*. 2016.

[Sug09] Tomomichi SUGIHARA. « Standing stabilizability and stepping maneuver in planar bipedalism based on the best COM-ZMP regulator ». In : *IEEE International Conference on Robotics and Automation*. 2009.