Completeness of Randomized Kinodynamic Planners with State-based Steering

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***Abstract*— The panorama of probabilistic completeness results for kinodynamic planners is still confusing. Most existing completeness proofs require strong assumptions that are difficult, if not impossible, to verify in practice. To make completeness results more useful, it is thus sensible to establish a classification of the various types of constraints and planning methods, and then attack each class with specific proofs and hypotheses that can be verified in practice. We propose such a classification, and provide a proof of probabilistic completeness for an important class of planners, namely those whose steering method is based on the interpolation of system trajectories in the state space. We also provide design guidelines for the interpolation function and discuss two criteria arising from our analysis: local boundedness and acceleration compliance.**

I. Introduction

A *deterministic* motion planning algorithm (or planner) is said to be *complete* if it returns a solution to a motion planning problem whenever one exists (see *e.g.*, [1]). A *randomized* planner is said to be *probabilistically complete* if the probability of returning a solution tends to one as execution time goes to infinity. The concepts of completeness and probabilistic completeness, although theoretical by nature, are also of practical interest: proving them requires one to specify what assumptions are needed for a planner to find solutions, *i.e.*, what types of problems can be solved. This provides more general guarantees than empirical results. Experiments can show that a planner works for a given combination of robot, environment, task, (set of tweaks and heuristics), but a proof of completeness is a certificate that the planner works for a whole set of problems, the size of this set being determined by the assumptions required to make the proof (the weaker the assumptions, the larger the set of solvable problems).

While the probabilistic completeness of randomized planners has been well established for systems with *geometric* constraints (such as obstacle avoidance), proofs for systems with *kinodynamic* constraints [2], [3], [4] have not yet reached the same level of generality: in many proofs, the assumptions made are quite strong and difficult to verify on practical systems (as a matter of fact, none of the previously mentioned works verified their hypotheses on non-trivial systems). One of the reasons for this lies in the large variety of kinodynamic constraints and planning methods that can be found.

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To make completeness proofs more useful in practice, it is thus necessary to estabish a classification of the different types of constraints and planning methods, and then attack each class with specific proofs and hypotheses that can be more easily checked. In section II, we propose such a classification based on kinodynamic constraints (non-holonomic or dynamics-bound-based) and planning methods (depending upon their underlying steering methods: analytic, control-based or state-based). We also discuss the shortcomings of existing completeness proofs. Then, in section III, we prove a completeness result for the class of state-based steering planners applied to systems with dynamics bounds. Finally, in section IV, we conclude by discussing the implications of our results as well as future research objectives.

II. Classification of Kinodyamic Constraints and of Steering Methods

A. Classification of Kinodynamic Constraints

Motion planning was first concerned only with *geometric* constraints such as obstacle avoidance or those imposed by the kinematic structures of manipulators [5], [6], [4], [2]. More recently, *kinodynamic* constraints, which stem from the *dynamical equations* the systems are subject to, have also been taken into account [7], [2], [8].

Kinodynamic constraints are more difficult to deal with than geometric constraints because they cannot in general be expressed using only *configuration-space variables* – such as the joint angles of a manipulator, the position and the orientation of a mobile robot, etc. They indeed involve *higher-order derivatives* of the configuration-space variables. However, these derivatives appear in the constraints in two main different ways, which involve different types of difficulties:

1) *Non-holonomic constraints* are non-integrable *equality* constraints on higher-order derivatives of the configuration-space variables. They can be first-order, as found in wheeled vehicles [9], or second-order, as found in under-actuated manipulators or space robots.
2) *Bounds on dynamic quantities* are *inequality* constraints on higher-order derivatives of the configuration-space variables. These include torque bounds for manipulators [10], ZMP constraints for walking robots, friction constraints in grasp synthesis, etc.

Some authors have considered systems that are subject to both types of constraints, such as under-actuated manipulators with torque bounds [11].

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We shall see in section II-C that control-based steering is more adapted to systems with non-holonomic constraints while state-based steering is more adapted to systems with inequality constraints.

B. Structure of Randomized Planners

Randomized planners such as such as Probabilistic Roadmaps (PRM) [6] or Rapidly-exploring Random Trees (RRT) [2] build a roadmap on the state space. In PRM, multiple samples are drawn from the free state space, and local steering is used to connect nearby states. Meanwhile, starting from an initial state x_{init}, RRT grows a tree by sampling random states and connecting them to their closest neighbor in the tree. Both behaviors are represented by the *extension* step, as given by Algorithm 1 below. It has the following sub-routines (see Fig. 1 for an illustration):

- *Sampling* $\mathrm{SAMPLE}(S)$: randomly samples from a set S;
- *Antecedent selection* $\mathrm{PARENTS}(x', V)$: returns a set of states x belonging to the roadmap V, from which steering towards x' will be attempted;
- *Local steering* $\mathrm{STEER}(x, x')$: tries to steer the system from x towards x'. If successful, returns a new node x_{steer} ready to be added to the roadmap. Depending on the planner, the successfulness criterion may *e.g.*, be "reach x' exactly" or "get close enough to x'".

The design of each sub-routine greatly impacts the quality and even the completeness of the resulting planner.

Algorithm 1 Extension step in randomized planners

Require: initial node x_{init}, number of iterations N

$(V, E) \leftarrow (\{x_{init}\}, \emptyset)$
for N steps **do**
  $x_{rand} \leftarrow \mathrm{SAMPLE}(\mathcal{X}_{free})$
  $X_{parents} \leftarrow \mathrm{PARENTS}(x_{rand}, V)$
  for x_{parent} in $X_{parents}$ **do**
    $x_{steer} \leftarrow \mathrm{STEER}(x_{parent}, x_{rand})$
    if x_{steer} is a valid state **then**
      $V \leftarrow V \cup \{x_{steer}\}$
      $E \leftarrow E \cup \{(x_{parent}, x_{steer})\}$
    end if
  end for
end for
return (V, E)

In the literature, SAMPLE is usually implemented as uniform random sampling. Some authors have suggested to use adaptive sampling to improve the performance of RRT or PRM planners.

In geometric planners, PARENTS is usually implemented by defining a metric (*e.g.*, the ℓ_2 norm) on the configuration space, and using nearest-neighbors as antecedents. Such a choice results in the so-called Voronoi bias of RRTs [2]. Both experiments and theoretical analysis support this choice for geometric planning. However, when moving to kinodynamic planning, designing a metric that yields good antecedents

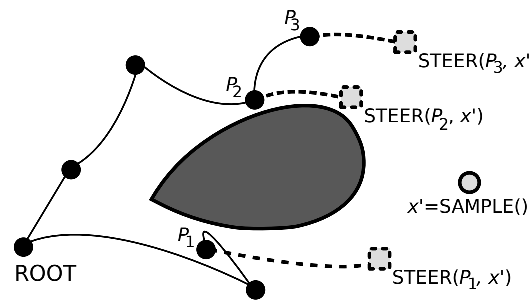

Fig. 1. Illustration of the extension routine of randomized planners. To grow the roadmap toward the sample x', the planner selects a number of parents $\mathrm{PARENTS}(x') = \{P_1, P_2, P_3\}$ from which it applies the $\mathrm{STEER}(P_i, x')$ method.

becomes as challenging as the motion planning problem itself, and the Euclidean norm becomes highly inefficient (see *e.g.*, [12] for an illustration in the case of the actuated pendulum subject to torque bounds).

The next section discusses the various implementations of the STEER sub-routine.

C. Classification of Steering Methods

We propose to classify existing steering methods into three categories. *Analytical steering* is the best method but is available only for a very limited class of systems. *State-based steering* is efficient but relies on inverse dynamics, and thus unapplicable to most systems with non-holonomic constraints. *Control-based* steering relies instead on forward dynamics and can thus be applied to a wider range of systems.

1) Control-based steering: compute a control function $u : [0, T] \to \mathcal{U}_{\mathsf{adm}}$ and generate the corresponding trajectory using *forward dynamics*. This approach does not guarantee that the desired state is reached by the end of the trajectory. In works such as [2], [8], random functions u are sampled from a family of primitives (*e.g.*, piecewise-constant functions), a number of them are tried and only the one bringing the system closest to the target is retained. Linear-Quadratic Regulation (LQR) [13], [14] is also control-based steering: in this case, u is computed as the optimal policy for a linear approximation of the system given a quadratic cost function.

2) State-based steering: interpolate a trajectory $\widetilde{\gamma} : [0, \Delta\widetilde{t}] \to \mathcal{C}$, for instance a Bezier curve matching the initial and target configurations and velocities, and compute a control that makes the system track it. For fully-actuated system, this is typically done using inverse dynamics. If no suitable control exists, the trajectory is rejected. Note that both the space $\mathrm{Im}(\widetilde{\gamma})$ and time $\tilde{\dot{\gamma}}(t)$ information of the interpolated trajectory have an effect on dynamic quantities, and thus the existence of admissible controls. More sophisticated methods such as Admissible Velocity Propagation [15] can be applied to restrict the interpolation to space, computing time and controls separately.

3) Analytical steering: with control-based steering, it is easy to respect differential constraints but difficult to reach the desired state. Conversely, with state-based steering, it is easy to reach the desired state but difficult to enforce differential constraints (for instance, inverse dynamics cannot always be used with non-holonomic systems). For some systems, steering functions satisfying both requirements are known, like Reeds and Shepp curves for cars. When it is the case, the problem can be reduced to path planning [9].

D. Previous Proofs of Probabilistic Completeness

Randomized planners such as Rapidly-exploring Random Trees (RTT) and Probabilistic Roadmaps (PRM) are popular because they are simple to implement yet efficient in practice. Proofs of probabilistic completeness come as another indicator in their favor. However, one should beware of general conceptions such as "RRT is probabilistically complete": as we will see, it is not always true for systems with kinodynamic constraints. Let us review completeness results that have been published for such systems.

Completeness of RRT planners has been established for *path planning* [2], [3], [4]. In their proof, Hsu et al. quantified the problem of narrow passages in configuration space with the notion of (α, β)-expansiveness [4]. The two constants α and β express a geometric lower bound on the rate of expansion of reachability areas. The authors later extended their solution to kinodynamic planning [8], using the same notion of expansiveness, but this time in the $\mathcal{X} \times \mathcal{T}$ (state and time) space with control-based steering. They established that, when $\alpha > 0$ and $\beta > 0$, their planner is probabilistically complete. However, whether $\alpha > 0$ or $\alpha = 0$ in the $\mathcal{X} \times \mathcal{T}$ space remains undiscussed, and the problem of evaluating (α, β) is deemed as difficult as the initial planning problem [4].

LaValle et al. provided a completeness argument for kinodynamic planning [2]. In their proof, they assumed the existence of an *attraction sequence*, which is a covering of the state space where two major problems of kinodynamic planning, namely steering and antecedent selection (see Section II-C), are already solved. However, conditions of existence of such a sequence are not discussed.

These two examples highlight our concern about completeness proofs: in both cases, probabilistic completeness is established under assumptions whose verification is at least as difficult as the motion planning problem itself. This observation does not question the quality of the associated planners, which have also been checked experimentally. Rather, it hints that too much of the complexity of kinodynamic planning has been abstracted into hypotheses. As a result, these completeness proof do not help us understand why these planners work (or don't work) in practice.

Karaman et al. introduced their path planning algorithm RRT* in [3] and extended it to kinodynamic planning with differential constraints in [16], providing a sketch of proof for the completeness of their solution. However, they assumed that their planner had access to the optimal cost metric and optimal local steering (wich means $\mathrm{STEER}(x_1, x_2)$ always returns the optimal trajectory starting from x_1 and ending at x_2), which restricts the analysis to systems for which these ideal solutions are known.

The same authors tackled the problem from a slightly different perspective in [17]. They now assumed that the PARENTS function computes *w-weighted boxes*, which are abstractions of the system's local controlability. It remains unclear to us how these boxes can be computed or approximated in practice, given that their definition involves the joint flow of vector fields spanning the tangent space of the system's manifold. Although their set of assumptions is of primary concern to us since we follow a similar approach in Section III, they did not prove their theorem, arguing that the reasoning was similar to the one in [3] for kinematic systems.

To the best of our knowledge, as of yet, there is no completeness proof for kinodynamic planners using state-based steering. We will establish such a result in the following section.

III. Completeness of State-based Steering Kinodynamic Planners

A. Terminology

A function is *smooth* when all its derivatives exist and are continuous. A function $f : A \to B$ between metric spaces is Lipschitz when there exists a constant K_f such that

$$\forall (x, y) \in A,\ \|f(x) - f(y)\| \leq K_f \|x - y\|.$$

Throughout the present paper, we will work within normed vector spaces and $\|\cdot\|$ will refer to the Euclidean norm $\|\cdot\|_2$. We will also consistently denote by K_f the (smallest possible) Lipschitz constant of any Lipschitz function f.

Let $\mathcal{C}$ denote n-dimensional configuration space, where n is the number of degrees of freedom of the robot. We will call *state space* the $2n$-dimensional manifold $\mathcal{X}$ of configuration and velocity coordinates. In the present paper, we only consider fully actuated systems. Let the control input space ("control space" for short) be an n-dimensional manifold $\mathcal{U}$. The dynamics of the robot follow the equations of motion, which can be written in generalized coordinates as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u. \tag{1}$$

Equivalently, the robot's dynamics follow the time-invariant differential system

$$\dot{x}(t) \ = \ f(x(t), u(t)), \tag{2}$$

where $x(t) \in \mathcal{X}$ and $u(t) \in \mathcal{U}$. We will assume that f is Lipschitz continuous in both of its arguments. The set $\mathcal{U}_{\mathsf{adm}}$ of admissible controls is assumed to be a compact subset of $\mathcal{U}$.

A trajectory is a continuous function $\gamma : [0, T] \to \mathcal{C}$. A path is the image of a trajectory. An admissible trajectory is a solution to the differential system (2). The kinematic motion planning problem is to find a path in the collision-free subset $\mathcal{C}_{free} \subset \mathcal{C}$ from an initial configuration q_{init} to any configuration q_{goal} in a set of goals. Meanwhile, the kinodynamic motion planning problem is to find an admissible trajectory from q_{init} to q_{goal}, both avoiding obstacles and following the system's dynamics.

A control function $t \mapsto u(t)$ is said to have δ-clearance when its image is in the δ-interior of the set of admissible controls, *i.e.*, for any time t, $\mathcal{B}(u(t), \delta) \subset \mathcal{U}_{\mathsf{adm}}$.

We define the distance between a state $x \in \mathcal{X}$ and the curve γ as:

$$\text{dist}_\gamma(x) := \min_{t \in [0,T]} \|(\gamma, \dot{\gamma})(t) - x\|$$

Whenever considering two states x and x', we will write $x =: (q, \dot{q})$ and $x' =: (q', \dot{q}')$. The prefix Δ will be used to denote variations between x and x', such as $\Delta x := x' - x$, $\Delta q := q' - q$, and so on. Similarly, for two time instants $t < t'$, we will write $\Delta t := t' - t$ and $\Delta g := g(t') - g(t)$ for any function g.

B. Assumptions for the Completeness Theorem

Our model for an $\mathcal{X}$-state randomized planner is given by Algorithm 1 using state-based steering. We make the following three assumptions on the system:

Assumption 1: The system is fully actuated.

Assumption 2: The set of admissible controls $\mathcal{U}_{\mathsf{adm}}$ is compact.

Assumption 3: The inverse of the differential constraint f from Equation (2), *i.e.*, the function f^{-1} s.t. $u = f^{-1}(x, \dot{x})$, is Lipschitz in both of its arguments.

Assumption 1 is a pre-requisite for the function f^{-1} used in Assumption 3 to be well-defined. The latter assumption is satisfied when f is given by the dynamics equations (1) as long as the matrices $M(q)$ and $C(q, \dot{q})$ have bounded norm, and the gravity term $g(q)$ is Lipschitz. Indeed, for a small displacement between x and x',

$$\begin{aligned} \|u' - u\| \quad \leq \quad & \|M\| \, \|\ddot{q}' - \ddot{q}\| + \|C(q, \dot{q})\| \, \|\dot{q}' - \dot{q}\| \\ & + K_g \, \|q' - q\| \end{aligned} \tag{3}$$

Regarding Assumption 2, since torque constraints are our main concern, we will make our proof of completeness for (note that the comparison is component-wise)

$$\mathcal{U}_{\mathsf{adm}} := \{u \in \mathcal{U}, \; |u| \leq \tau_{\mathsf{max}}\},$$

which is indeed compact. The generalization to an arbitrary compact set presents no technical difficulty.

Let us now turn to the design of the interpolation routine. We make the following three hypotheses:

Assumption 4: Interpolated trajectories $\widetilde{\gamma}$ are smooth Lipschitz functions, and their time-derivatives $\dot{\widetilde{\gamma}}$ (*i.e.*, interpolated velocities) are also Lipschitz.

Assumption 5 (Local boundedness): We suppose that there exists a constant η such that, for any $(x, x') \in \mathcal{X}^2$, the interpolated trajectory $\widetilde{\gamma}: \; [0, \Delta\widetilde{t}] \to \mathcal{C}$ between x and x' is included in a ball of center x and radius $\eta \, \|x' - x\|$.

Assumption 6 (Acceleration compliance): The acceleration of interpolated trajectories uniformly converges to the discrete velocity derivative, *i.e.*, there exists some $\nu > 0$ such that, if $\widetilde{\gamma}: [0, \Delta\widetilde{t}] \to \mathcal{C}$ results from $\text{INTERPOLATE}(x, x')$, then

$$\forall \tau \in [0, \Delta\widetilde{t}], \; \left\| \ddot{\widetilde{\gamma}}(\tau) - \frac{\|\dot{q}\|}{\|\Delta q\|} \Delta \dot{q} \right\| \; \leq \; \nu \, \|\Delta x\|$$

Assumption 4 is quite easy to satisfy. Assumption 5 bounds the position and velocity of interpolated trajectories with respect to the neighborhood of x and x', while Assumption 6 bounds their acceleration with respect to the discrete derivative of the velocity between x and x'. These three assumptions are design guidelines for the interpolation routine. They ensure that the resulting local planner will always look for smaller trajectories when working in smaller neighborhoods. Note that we consider fully-actuated (thus small-space controllable) systems for which such solutions always exist.

C. Verifying the Assumptions on the Double Pendulum

To illustrate the practicality of these assumptions, let us consider the standard example of a fully-actuated double pendulum under torque constraints.

1) System assumptions: When pendulum links have mass m and length l, the gravity term $g(\theta_1, \theta_2) = \frac{mgl}{2} [\sin\theta_1 + \sin(\theta_1 + \theta_2) \;\; \sin(\theta_1 + \theta_2)]$ is Lipschitz with constant $K_g = 2mgl$. Meanwhile, the inertial term is bounded by $\|M\| \leq 3ml^2$ and, when joint angular velocities are bounded by ω, the norm of the Coriolis tensor is bounded by $2\omega ml^2$. Therefore, from Equation (3), there exist a Lipschitz constant $K_{f^{-1}}$.

2) Interpolation: A simple second-order polynomial interpolation is given by:

$$\gamma(t) = \frac{\Delta \dot{q}}{2\Delta t} t^2 + \left(\frac{\Delta q}{\Delta t} - \frac{\Delta \dot{q}}{2} \right) t + q, \tag{4}$$

where $\Delta t := \frac{\|\Delta q\|}{\|\dot{q}\|}$. This expression only matches position and acceleration constraints (in particular, it does not work when $\|\dot{q}\| = 0$). One can use higher-order polynomials in a similar fashion to take velocities into account as well. All polynomials satisfy the smoothness Assumption 4. Meanwhile, Assumption 6 is verified as the dominating term in (4) is exactly the discrete velocity time-derivative. Finally, one can check with no computational hassle that $\|\gamma(t) - \gamma(0)\| \leq (1 + \|\Delta \dot{q}\| / \|\dot{q}\|) \, \|\Delta q\| \to 0$ when $\|\Delta x\| \to 0$.

D. Completeness Theorem

We can now state our main result:

Theorem 1: Consider a time-invariant differential system (2) with Lipschitz-continuous f and full actuation over a compact set of admissible controls $\mathcal{U}_{\mathsf{adm}}$. Suppose that the kinodynamic planning problem between two states x_{init} and x_{goal} admits a smooth Lipschitz solution $\gamma: [0, T] \to \mathcal{C}$ with δ-clearance in control space. Let $\mathcal{K}$ denote a randomized motion planner (Algorithm 1) using state-based steering and a locally bounded, Lipschitz, acceleration-compliant interpolation primitive. $\mathcal{K}$ is probabilistically complete.

Let us start the proof of this theorem with three lemmas. Detailed proofs of these lemmas are provided in the supplementary material.[1]

[1] `https://scaron.info/research/icra-2014.html`

Lemma 1: Let $g : [0, T] \to \mathbb{R}^k$ denote a smooth Lipschitz function. Then, for any $(t, t') \in [0, T]^2$,

$$\left\| \dot{g}(t) - \frac{g(t') - g(t)}{|t' - t|} \right\| \;\leq\; \frac{K_g}{2} |t' - t|.$$

Lemma 2: If there exists a trajectory γ with δ-clearance in control space, then there exists $\delta' < \delta$ and a trajectory γ' with δ'-clearance in control space such that $\inf_t \|\ddot{\gamma}'(t)\| > 0$.

Lemma 3: If there exists a trajectory γ with δ-clearance in control space, then there exists $\delta' < \delta$ and a trajectory γ' with δ'-clearance in control space such that $\inf_t \|\dot{\gamma}'(t)\| > 0$.

Let $\gamma : [0, T] \to \mathcal{C}, t \mapsto \gamma(t)$ denote a smooth Lipschitz admissible trajectory from x_{init} to x_{goal} with δ-clearance in control space. We define:

$$\left\{ \begin{array}{lcl} \dot{M} & := & \max_t \|\dot{\gamma}(t)\| \\ \dot{m} & := & \min_t \|\dot{\gamma}(t)\| \end{array} \right. \qquad \left\{ \begin{array}{lcl} \ddot{M} & := & \max_t \|\ddot{\gamma}(t)\| \\ \ddot{m} & := & \min_t \|\ddot{\gamma}(t)\| \end{array} \right.$$

From lemmas 2 and 3, we can suppose without loss of generality that $\dot{m} > 0$ and $\ddot{m} > 0$. Consider two states x and x' and the corresponding time instants on the trajectory

$$\left\{ \begin{array}{lcl} t & := & \arg\min_t \|(\gamma(t), \dot{\gamma}(t)) - x\| \,, \\ t' & := & \arg\min_t \|(\gamma(t), \dot{\gamma}(t)) - x'\| \,. \end{array} \right.$$

We can suppose w.l.o.g. that $t < t'$. First, note that there exists $\delta t_1 > 0$ such that, for any $\Delta t \leq \delta t_1$,

$$\frac{\|\Delta\gamma\|}{\Delta t} \geq \frac{\dot{m}}{2}, \quad \frac{\|\Delta\dot{\gamma}\|}{\Delta t} \geq \frac{\ddot{m}}{2}, \quad \frac{\|\Delta\dot{\gamma}\|}{\|\Delta\gamma\|} \leq \frac{2\ddot{M}}{\dot{m}}.$$

Indeed, the three functions $\Delta t \mapsto \frac{\|\Delta\gamma\|}{\Delta t}$, $\Delta t \mapsto \frac{\|\Delta\dot{\gamma}\|}{\Delta t}$ and $\Delta t \mapsto \frac{\|\Delta\dot{\gamma}\|}{\|\Delta\gamma\|}$. are continuous over the compact set $[0, T]$, hence uniformly continuous, and their limits when $\Delta t \to 0$ are respectively $\|\dot{\gamma}(t)\| \geq \dot{m}$, $\|\ddot{\gamma}(t)\| \geq \ddot{m}$ and $\frac{\|\ddot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|} \leq \frac{\ddot{M}}{\dot{m}}$. In what follows, we will then suppose that Δt is smaller than this first threshold δt_1.

Let $\widetilde{\gamma} : [0, \Delta\widetilde{t}] \to \mathcal{C}$ denote the result of INTERPOLATE(x, x'). For $\tau \in [0, \Delta\widetilde{t}]$, the torque required to follow the trajectory $\widetilde{\gamma}$ is $\widetilde{u}(\tau) := f(\widetilde{\gamma}(\tau), \dot{\widetilde{\gamma}}(\tau), \ddot{\widetilde{\gamma}}(\tau))$. Since Im$(u) \subset$ int$_\delta(\mathcal{T})$,

$$\begin{aligned} |\widetilde{u}(\tau)| \;&\leq |\widetilde{u}(\tau) - u(t)| + |u(t)| \\ &\leq \left| f(\widetilde{\gamma}(\tau), \dot{\widetilde{\gamma}}(\tau), \ddot{\widetilde{\gamma}}(\tau)) - f(\gamma(t), \dot{\gamma}(t), \ddot{\gamma}(t)) \right| \\ &\quad + (1 - \delta)\, \tau_{\mathsf{max}}, \end{aligned}$$

where the comparison here is component-wise. If the first term in this upper bound is $\leq \delta\, \tau_{\mathsf{max}}$, then the system will be able to track $\widetilde{\gamma}$ at time τ. We can rewrite it as follows:

$$\begin{aligned} &\left| f(\widetilde{\gamma}(\tau), \dot{\widetilde{\gamma}}(\tau), \ddot{\widetilde{\gamma}}(\tau)) - f(\gamma(t), \dot{\gamma}(t), \ddot{\gamma}(t)) \right| \\ &\leq \left\| f(\widetilde{\gamma}(\tau), \dot{\widetilde{\gamma}}(\tau), \ddot{\widetilde{\gamma}}(\tau)) - f(\gamma(t), \dot{\gamma}(t), \ddot{\gamma}(t)) \right\|_\infty \\ &\leq\; K_f \left\| (\widetilde{\gamma}(\tau), \dot{\widetilde{\gamma}}(\tau)) - (\gamma(t), \dot{\gamma}(t)) \right\| + K_f \left\| \ddot{\widetilde{\gamma}}(\tau) - \ddot{\gamma}(t) \right\| \\ &\leq \underbrace{K_f \left[(\eta + \nu) \|\Delta x\| + \mathrm{dist}_\gamma(x) \right]}_{\text{distance term (D)}} + \underbrace{K_f \left\| \frac{\|\dot{q}\|}{\|\Delta q\|} \Delta\dot{q} - \ddot{\gamma}(t) \right\|}_{\text{acceleration term (A)}}, \quad (\star) \end{aligned}$$

where we used the triangular inequality, the Lipschitz condition on f, as well as local boundedness (Assumption 5) and acceleration compliance (Assumption 6) of the interpolated trajectory. The transition from the norm $\|\cdot\|_\infty$ to $\|\cdot\|$ is possible because all norms of $\mathbb{R}^n$ are equivalent (a change in norm will be reflected by a different constant K_f).

1) Bounding the acceleration term: the discrete velocity derivative $\frac{\|\dot{q}\|}{\|\Delta q\|} \Delta\dot{q}$ can be further decomposed into:

$$\begin{aligned} \left\| \frac{\|\dot{q}\|}{\|\Delta q\|} \Delta\dot{q} - \ddot{\gamma}(t) \right\| \;&\leq\; \left\| \Delta\dot{q} \frac{\|\dot{q}\|}{\|\Delta q\|} - \Delta\dot{\gamma} \frac{\|\dot{\gamma}(t)\|}{\|\Delta\gamma\|} \right\| \\ &+ \frac{\|\Delta\dot{\gamma}\|}{\|\Delta\gamma\|} \left| \|\dot{\gamma}(t)\| - \frac{\|\Delta\gamma\|}{\Delta t} \right| + \left\| \frac{\Delta\dot{\gamma}}{\Delta t} - \ddot{\gamma}(t) \right\|. \end{aligned}$$

Let us call these three terms (A1), (A2) and (A3). From Lemma 1, (A3) $\leq \frac{K_{\dot{\gamma}}}{2} \Delta t$ and

$$\text{(A2)} \;\leq\; \frac{K_{\dot{\gamma}}}{2} \frac{\|\Delta\dot{\gamma}\|}{\|\Delta\gamma\|} \Delta t \;\leq\; \frac{K_{\dot{\gamma}} \ddot{M}}{\dot{m}} \Delta t.$$

Then, defining $\delta t_2 := \min\left(\delta t_1, \frac{\delta\tau_{\mathsf{max}}}{2K_{\dot{\gamma}}}, \frac{\delta\dot{m}\tau_{\mathsf{max}}}{4\ddot{M}K_{\dot{\gamma}}} \right)$, we have that, for any $\Delta t < \delta t_2$, (A2) and (A3) are upper bounded by $\frac{\delta\tau_{\mathsf{max}}}{4K_f}$. The expression $\Delta\dot{q}\frac{\|\dot{q}\|}{\|\Delta\dot{q}\|}$ in (A1) represents the discrete derivative of the velocity $\dot{q}$ between q and q' (its continuous analog would be $\frac{\|\dot{q}\|\,\mathrm{d}\dot{q}}{\|\dot{q}\|\,\mathrm{d}t} = \frac{\mathrm{d}\dot{q}}{\mathrm{d}t}$). Thus, (A1) can be seen as the deviation between the discrete accelerations of $\widetilde{\gamma}$ and γ. Let us decompose this expression in terms of norm and angular deviation: (A1) is less than

$$\left\| \left(\frac{\Delta\dot{\gamma}}{\|\Delta\dot{\gamma}\|} - \frac{\Delta\dot{q}}{\|\Delta\dot{q}\|} \right) \frac{\|\dot{\gamma}\|\,\|\Delta\dot{\gamma}\|}{\|\Delta\gamma\|} + \frac{\Delta\dot{q}}{\|\Delta\dot{q}\|} \left(\frac{\|\Delta\dot{\gamma}\|\,\|\dot{\gamma}\|}{\|\Delta\gamma\|} - \frac{\|\Delta\dot{q}\|\,\|\dot{q}\|}{\|\Delta q\|} \right) \right\|$$

that is,

$$2\frac{\|\dot{\gamma}\|\,\|\Delta\dot{\gamma}\|}{\|\Delta\gamma\|} \underbrace{\left(1 - \cos(\widehat{\Delta\dot{q}, \Delta\dot{\gamma}}) \right)}_{\text{angular deviation term } (\theta)} + \underbrace{\left| \frac{\|\dot{\gamma}\|\,\|\Delta\dot{\gamma}\|}{\|\Delta\gamma\|} - \frac{\|\Delta\dot{q}\|\,\|\dot{q}\|}{\|\Delta q\|} \right|}_{\text{norm deviation term (N)}}$$

Since the factor $\frac{2\|\dot{\gamma}\|\|\Delta\dot{\gamma}\|}{\|\Delta\gamma\|}$ before the angular deviation (θ) is bounded by $\frac{4\dot{M}\ddot{M}}{\dot{m}}$, $(\widehat{\Delta\dot{q}, \Delta\dot{\gamma}}) \to 0$ is a sufficient condition for $(\theta) \to 0$. We will show that both the norm and angular deviation terms tend to zero as $\Delta t \to 0$.

2) Bounding the norm (N): let us suppose that $\mathrm{dist}_\gamma(x)$ and $\mathrm{dist}_\gamma(x')$ are $\leq \frac{1}{2}\dot{m}\Delta t^2 =: \delta\rho$. We can expand (N) as follows:

$$\begin{aligned} \text{(N)} \;&\leq\; \frac{\|\Delta\dot{\gamma}\|}{\|\Delta\gamma\|} \left| \|\dot{\gamma}\| - \|\dot{q}\| \right| + \|\dot{q}\| \left| \frac{\|\Delta\dot{\gamma}\|}{\|\Delta\gamma\|} - \frac{\|\Delta\dot{q}\|}{\|\Delta q\|} \right| \\ &\leq\; \frac{2\ddot{M}}{\dot{m}}\delta\rho + \frac{\|\dot{q}\|}{\|\Delta\gamma\|\,\|\Delta q\|} \left| \|\Delta\dot{\gamma}\|\,\|\Delta q\| - \|\Delta q\|\,\|\Delta\dot{q}\| \right| \\ &\leq\; \frac{2\ddot{M}}{\dot{m}}\delta\rho + \frac{\|\dot{q}\|\,(\|\Delta\gamma\| + \|\Delta\dot{\gamma}\|)\delta\rho}{\|\Delta\gamma\|\,\|\Delta q\|} \\ &\leq\; \frac{2\ddot{M}}{\dot{m}}\delta\rho + \delta\rho\frac{\|\dot{q}\|}{\|\Delta q\|} \left[1 + \frac{2\ddot{M}}{\dot{m}} \right] \\ &\leq\; \frac{2\ddot{M}}{\dot{m}}\delta\rho + \delta\rho\frac{\|\dot{\gamma}\| + \delta\rho}{\|\Delta\gamma\| - \delta\rho} \left[1 + \frac{2\ddot{M}}{\dot{m}} \right] \end{aligned}$$

$$\leq \left[\ddot{M}\Delta t + \frac{(\dot{m} + 2\ddot{M})(2\dot{M} + \dot{m}\Delta t^2)}{\dot{m}^2(1 - \Delta t)} \right] \Delta t$$

This last bound is expressed only in terms of Δt and constants $\dot{m}, \dot{M}$ and $\ddot{M}$. Since it tends to zero as $\Delta t \to 0$, there exists some duration $\delta t_3 \leq \delta t_2$ such that, for any $\Delta t \leq \delta t_3$, (N) $\leq \frac{\delta \tau_{\mathsf{max}}}{8K_f}$.

3) Bounding the angular deviation: simple vector geometry shows that

$$\sin(\widehat{\Delta\dot{q}, \Delta\dot{\gamma}}) \leq \frac{\text{dist}_\gamma(x) + \text{dist}_\gamma(x')}{\|\Delta\dot{\gamma}\|} \leq \frac{\delta\rho}{\ddot{m}\Delta t} \leq \frac{\dot{m}}{2\ddot{m}}\Delta t.$$

Since $1 - \cos\theta < \sin\theta$ for any $\theta \in [0, \pi/2]$, there exists a duration $\delta t_4 \leq \delta t_3$ such that $\Delta t < \delta t_4 \Rightarrow (\theta) \leq \frac{\delta \tau_{\mathsf{max}}}{8K_f}$. Combining our bounds on terms (A2), (A3), (N) and (θ), we have showed so far that, when Δt is small enough, the acceleration term is upper bounded by $\frac{3}{4}\delta\tau_{\mathsf{max}}$.

4) Bounding the distance term (D): the remaining term is proportional to

$$\begin{aligned}(\eta + \nu)\|\Delta x\| + \text{dist}_\gamma(x) &\leq (2\delta\rho + \|\Delta\gamma\|)(\eta + \nu) + \delta\rho \\ &\leq \frac{K_\gamma(\eta + \nu) + 3\dot{m}\Delta t}{2}\Delta t\end{aligned}$$

Hence, there exist a final $\delta t \leq \delta t_4$ such that, when $\Delta t < \delta t$, this last bound becomes $\leq \frac{\delta \tau_{\mathsf{max}}}{4K_f}$ as well. Combining all our bounds, we have established the existence of a duration δt such that $\Delta t \leq \delta t \Rightarrow |\widetilde{u}(\tau)| \leq \tau_{\mathsf{max}}$.

5) Link with completeness: let us summarize our reasoning so far. We have iteratively constructed a duration δt and a radius $\delta\rho$, independent from t or t', such that, as soon as $|t' - t| < \delta t$, $\text{dist}_\gamma(x) < \delta\rho$ and $\text{dist}_\gamma(x') < \delta\rho$, the system can track the trajectory $\text{INTERPOLATE}(x, x')$.

The proof of completeness of the whole randomized planner follows directly from this construction. Let us denote by $\mathcal{B}_t := \mathcal{B}((\gamma, \dot{\gamma})(t), \delta\rho)$, the ball of radius $\delta\rho$ centered on $(\gamma, \dot{\gamma})(t) \in \mathcal{X}$. Suppose that the roadmap contains a state $x \in \mathcal{B}_t$, and let $t' := \min(T, t + \delta t)$. If the planner samples a state $x' \in \mathcal{B}_{t'}$, the interpolation between x and x' will be successful and x' will be added to the roadmap. Since the volume of $\mathcal{B}_{t'}$ is non-zero for the Lebesgue metric, the event $\{\text{SAMPLE}(\mathcal{X}_{\mathit{free}}) \in \mathcal{B}_{t'}\}$ will happen with probability one as the number of extensions goes to infinity.

At the initialization of the planner, the roadmap is reduced to $x_{\mathit{init}} = (\gamma(0), \dot{\gamma}(0))$. Therefore, using the property above, by induction on the number of time steps δt, the last state $(\gamma(T), \dot{\gamma}(T))$ will be eventually added to the roadmap with probability one, which establishes the probabilistic completeness of the randomized planner. □

IV. Conclusion

The goal of the present paper was to clarify the panorama of completeness results in randomized kinodynamic planning. We noted that existing proofs usually rely on assumptions too strong to be verified on practical systems. We proposed a classification of the various types of kinodynamic constraints and planning methods used in the field, and went on to prove probabilistic completeness for an important class of planners, namely those which steer by interpolating system trajectories in the state space. Along the way, our analysis also provided some insights into the design of such interpolation functions.

The proof strategy that we used, *i.e.*, the inclusion of the solution trajectory into a "tube" of non-zero volume, is not new. It is related to the "attraction sequence" hypothesized in [2], and can be traced back to seminal papers such as [7]. However, to the best of our knowledge, our work is the first theoretical analysis to establish the existence and explicitely construct such a bounding tube. This construction is an extra link with reality: for a given system, one can actually check for full actuation, compacity of the control set and Lipschitz continuity of the dynamics function. Similarly, when designing her interpolation function, one can easily check for properties such as local boundedness and acceleration compliance.

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