Supplementary Material for the paper "Completeness of Randomized Kinodynamic Planners with State-based Steering"

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January 28, 2014

1 Proof of the Lemmas

Our first lemma upper-bounds the difference between the variation rate and derivative of a Lipschitz function.

Lemma 1. *Let* $g : [0,T] \to \mathbb{R}^k$ *denote a smooth Lipschitz function. Then, for any* $(t,t') \in [0,T]^2$,

$$\left\| \dot{g}(t) - \frac{g(t') - g(t)}{|t' - t|} \right\| \leq \frac{K_g}{2} |t' - t|.$$

Proof. Let $t' > t$. Then,

$$\begin{aligned}
\left\| \dot{g}(t) - \frac{g(t') - g(t)}{t' - t} \right\| &\leq \frac{1}{t'-t} \left\| \int_t^{t'} (\dot{g}(t) - \dot{g}(w))\, \mathrm{d}w \right\| \\
&\leq \frac{1}{t'-t} \int_t^{t'} \|\dot{g}(t) - \dot{g}(w)\|\, \mathrm{d}w \\
&\leq \frac{K_g}{t'-t} \int_t^{t'} |t - w|\, \mathrm{d}w \\
&\leq \frac{K_g}{2}(t' - t).
\end{aligned}$$

□

Lemma 2. *If there exists a trajectory* γ *with* δ*-clearance in control space, then there exists* $\delta' < \delta$ *and a trajectory* γ' *with* δ'*-clearance in control space such that* $\inf_t \|\ddot{\gamma}'(t)\| > 0$.

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Sketch of proof. If there is a time interval $[t, t']$ on which $\ddot{\gamma} \equiv 0$, one can leverage full actuation and δ-clearance in control to increment each coordinate with a small wave function $\delta\ddot{\gamma}_i$ of amplitude $\delta\ddot{q}_i$ and zero integral over $[t, t']$. The amplitude $\delta\ddot{q}_i$ is chosen so as to guarantee δ'-clearance in control space, for some $\delta' < \delta$.[1]

We can therefore assume that w.l.o.g. that the roots of $\ddot{\gamma}$ form a discrete set. Let t_0 be such a root. Again, δ-clearance in control and full actuation can be leveraged into adding a small perturbation $\delta\ddot{\gamma}_i$ to each coordinate around t_0. To ensure that $\ddot{\gamma}(t_0)$ becomes non-zero without creating new roots at other time instants, one needs to ensure that the coordinate perturbations are not time-correlated, which is easy to do, for instance using sine waves with different different periods. Special care needs to be taken if the root is at the first (or last) time instants of the trajectory. However, since we do not require accelerations (nor controls) to be continuous, one can simply shift the wave so as to start with (resp. end on) a non-zero value. ☐

Lemma 3. *If there exists a trajectory γ with δ-clearance in control space, then there exists $\delta' < \delta$ and a trajectory γ' with δ'-clearance in control space such that* $\inf_t \|\dot{\gamma}'(t)\| > 0$.

Sketch of proof. The argument is the same as in the proof for Lemma 2: add a small perturbation wave of controlled amplitude to the velocity coordinates. However, the system is controlled in acceleration and not velocity. To overcome this, one can use sine waves as a basis family for the perturbations: their derivatives are cosine waves of controlled amplitude, which can be added to the acceleration coordinates using full actuation and reducing the δ-clearance in control. Boundary values for these perturbations will be non-zero, which is not a problem since we do not require acceleration nor control to be continuous. ☐

[1] In the presence of $\mathcal{C}$-space obstacles or velocity limits, one can refine this wave as $\delta\ddot{\gamma}_i(w) = \delta\ddot{q}_i \sin\left(\frac{kw}{t'-t}\right)$, where the period $\frac{t'-t}{k}$ is chosen so as to bound the deviation in velocity and position incurred by the perturbation.