Supplementary material for "Balance control using both ZMP and COM height variations: a convex boundedness approach"

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This document provides a derivation of the viability conditions given in Section III.A of the paper [1]. To shorten notations, we will denote by:

$$a \stackrel{\text{def}}{=} \sqrt{\lambda_{\min}} \qquad\qquad b \stackrel{\text{def}}{=} \sqrt{\lambda_{\max}} \tag{1}$$

Values of all quantities at time $t_\text{i} = 0$ are denoted by the $\square_\text{i}$ subscript.

1 Viability when $\lambda \geq a^2$

Recall that the damping function ω is the non-negative finite solution[1] of $\dot{\omega} = \omega^2 - \lambda$. Then, the inequality $\lambda \geq a^2$ implies that:

$$\dot{\omega} \leq \omega^2 - a^2 = (\omega - a)(\omega + a) \tag{2}$$

Note how, if $\omega_\text{i} < a$ then $\limsup_{t \to +\infty} \omega(t) \leq -a$, which implies that $\omega_\text{i} \geq a$ is a necessary condition for viability. Next, when $\omega_\text{i} > a$, we can solve the above differential inequality by comparing to its upper-bounding profile:

$$\dot{\overline{\omega}} = (\overline{\omega} - a)(\overline{\omega} + a) \qquad\qquad \overline{\omega}_\text{i} = \omega_\text{i} \tag{3}$$

We can solve the differential equation on the upper-bounding profile as:

$$\frac{\dot{\overline{\omega}}}{(a - \overline{\omega})(\overline{\omega} + a)} = -1 \iff \frac{\dot{\overline{\omega}}}{a - \overline{\omega}} + \frac{\dot{\overline{\omega}}}{a + \overline{\omega}} = -2a \iff \left(\log \frac{\overline{\omega} + a}{a - \overline{\omega}}\right) = -2a \tag{4}$$

So that, *in fine*:

$$\omega(t) \leq \overline{\omega}(t) = a \frac{1 - \frac{a - \omega_\text{i}}{a + \omega_\text{i}} e^{2at}}{1 + \frac{a - \omega_\text{i}}{a + \omega_\text{i}} e^{2at}} = -a \frac{\cosh(a(T_a - t))}{\sinh(a(T_a - t))} \tag{5}$$

where $T_a \stackrel{\text{def}}{=} \frac{-1}{2a} \log \frac{\omega_\text{i} - a}{\omega_\text{i} + a}$. This inequality implies that, for all $t \leq T_a$,

$$\Omega(t) \leq \log \sinh(aT_a) - \log \sinh(a(T_a - t)) \tag{6}$$

Injecting this upper bound in the boundedness condition yields:

$$\xi_\text{i} \geq \frac{g}{\sinh(aT_a)} \int_\text{i}^{T_a} \sinh(a(T_a - t)) \mathrm{d}t = \frac{g}{a \sinh(aT_a)} (\cosh(aT_a) - 1) \tag{7}$$

Let us now denote by $X \stackrel{\text{def}}{=} \sqrt{\frac{\omega_\text{i} - a}{\omega_\text{i} + a}} \in [0, 1]$, so that:

$$\frac{\cosh(aT_a) - 1}{\sinh(aT_a)} = \frac{X + X^{-1} - 2}{X^{-1} - X} = \frac{(1 - X)^2}{1 - X^2} = \frac{(1 - X)^2}{1 - \frac{\omega_\text{i} - a}{\omega_\text{i} + a}} = \frac{(1 - X)^2}{2a} (\omega_\text{i} + a) \tag{8}$$

[1] See also Section II of [2] for more details on the properties expected from ω.

We can now rewrite our viability condition (7) as:

$$\xi_{\mathrm{i}} \geq \frac{g}{2a^2}(\omega_{\mathrm{i}} + a)\left(1 - \sqrt{\frac{\omega_{\mathrm{i}} - a}{\omega_{\mathrm{i}} + a}}\right)^2 \tag{9}$$

$$= \frac{g}{2a^2}\left(\sqrt{\omega_{\mathrm{i}} + a} - \sqrt{\omega_{\mathrm{i}} - a}\right)^2 \tag{10}$$

$$= \frac{g}{a^2}(\omega_{\mathrm{i}} - \sqrt{\omega_{\mathrm{i}}^2 - a^2}) \tag{11}$$

Recalling that $\omega_{\mathrm{i}} = -\dot{x}_{\mathrm{i}}/x_{\mathrm{i}}$, this condition can finally be rewritten as:

$$\dot{z}_{\mathrm{i}} - \frac{\dot{x}_{\mathrm{i}}}{x_{\mathrm{i}}} z_{\mathrm{i}} - \frac{g}{a^2}\left(-\frac{\dot{x}_{\mathrm{i}}}{x_{\mathrm{i}}} + \sqrt{\frac{(\dot{x}_{\mathrm{i}})^2}{x_{\mathrm{i}}^2} - a^2}\right) \geq 0 \tag{12}$$

One could also invert this equation to obtain the largest value of a for which solutions exist.

2 Viability when $a^2 \leq \lambda \leq b^2$

Similarly to the setting where $\lambda \geq a^2$, the condition $\lambda < b^2$ prevents the damping ω from taking values larger than b: if $\omega_{\mathrm{i}} > b$, then

$$\omega(t) \geq -b\frac{\cosh(b(T_b - t))}{\sinh(b(T_b - t))} \tag{13}$$

$$\Omega(t) \geq \log\sinh(bT_b) - \log\sinh(b(T_b - t)) \tag{14}$$

This means that, at a time no greater than T_b, $x(t)$ will reach 0. Yet, at this finite instant, $\dot{x}(t)$ will be positive and the robot will overshoot its target static equilibrium. Consequently, a necessary condition for viability is that:

$$a \leq \omega(t) = \frac{\dot{x}(t)}{x(t)} \leq b \tag{15}$$

2.1 Updated lower bound

We need first to re-assess the lower-bound on ξ_{i} that we obtained previously. To minimize $\int e^{-\Omega(t)}\mathrm{d}t$, $\lambda(t)$ must be equal to a^2 until the time T_1 such that $\omega(T_1) = b$, at which point it will switch to $\lambda(t) = b^2$ for all $t \geq T_1$. Let us denote by $\overline{\omega}$ the corresponding solution to $\dot{\omega} = \omega^2 - \lambda$, and $\overline{\Omega}$ its antiderivative such that $\overline{\Omega}_{\mathrm{i}} = 0$. For any $t \leq T_1$,

$$\overline{\omega}(t) = -\frac{a}{\tanh(a(T_a - t))} \qquad e^{-\overline{\Omega}(t)} = \frac{\sinh(a(T_a - t))}{\sinh(aT_a)} \tag{16}$$

In particular, the value of the latter expression at time T_1 is given by:

$$T_1 = T_a - \frac{1}{a}\mathrm{artanh}\left(\frac{a}{b}\right) \qquad e^{-\Omega(T_1)} = \frac{\sinh(\mathrm{artanh}(a/b))}{\sinh(aT_a)} = \frac{a}{\sinh(aT_a)\sqrt{b^2 - a^2}} \tag{17}$$

From there, we deduce a lower bound on the boundedness time integral:

$$\int_0^\infty e^{-\Omega(t)}\mathrm{d}t \geq \int_0^\infty e^{-\overline{\Omega}(t)}\mathrm{d}t \tag{18}$$

$$= \int_0^{T_1} \frac{\sinh(a(T_a - t))}{\sinh(aT_a)}\mathrm{d}t + \frac{\sinh(a(T_a - T_1))}{\sinh(aT_a)}\int_{T_1}^\infty e^{b(T_1 - t)}\mathrm{d}t \tag{19}$$

$$= \frac{\cosh(aT_a) - \cosh(a(T_a - T_1))}{a\sinh(aT_a)} + \frac{\sinh(a(T_a - T_1))}{b\sinh(aT_a)} \tag{20}$$

$$= \frac{1}{a\tanh(aT_a))} - \frac{b/a - a/b}{\sqrt{b^2 - a^2}\sinh(aT_a)} \tag{21}$$

2.2 New upper bound

In a fashion similar to the previous paragraph, to make ω as small as possible, $\lambda(t)$ must be equal to b^2 until some time T_2 such that $\omega(T_2) = a$, at which point it will switch to $\lambda(t) = a^2$ for all $t \geq T_2$. Let us denote by $\underline{\omega}$ the corresponding solution to $\dot{\omega} = \omega^2 - \lambda$, and $\underline{\Omega}$ its antiderivative such that $\underline{\Omega}_{\mathrm{i}} = 0$. For any $t \leq T_2$,

$$\underline{\omega}(t) = b\frac{1 - \frac{b-\omega_{\mathrm{i}}}{b+\omega_{\mathrm{i}}}e^{2bt}}{1 + \frac{b-\omega_{\mathrm{i}}}{b+\omega_{\mathrm{i}}}e^{2bt}} \tag{22}$$

Let us denote by $Y \stackrel{\text{def}}{=} \sqrt{\frac{b-\omega_{\mathrm{i}}}{b+\omega_{\mathrm{i}}}}$ and $T_b \stackrel{\text{def}}{=} -(\log Y)/b$. Then,

$$\underline{\omega}(t) = b\tanh(b(T_b - t)) \qquad\qquad e^{-\underline{\Omega}(t)} = \frac{\cosh(b(T_b - t))}{\cosh(bT_b)} \tag{23}$$

Let us further denote by $T_2 = \inf\{t : \underline{\omega}(t) = a\} = T_b - \frac{\operatorname{argth}(a/b)}{b}$. Then,

$$\begin{aligned}
\int_0^{\infty} e^{-\Omega(t)}\mathrm{d}t &\leq \int_0^{T_1} e^{-Y\underline{\Omega}(t)}\mathrm{d}t + e^{-\underline{\Omega}(T_1)}\int_0^{+\infty} e^{-at}\mathrm{d}t && (24)\\
&= \frac{\sinh(bT_b) - \sinh(b(T_b - T_1))}{b\cosh(bT_b)} + \frac{\cosh(b(T_b - T_1))}{a\cosh(bT_b)} && (25)\\
&= \frac{\tanh(bT_b)}{b} + \frac{b/a - a/b}{\sqrt{b^2 - a^2}\cosh(bT_b)} && (26)
\end{aligned}$$

2.3 Final viability condition

Combining inequalities (21)–(26) with, as before, the relationship $\omega_{\mathrm{i}} = -\dot{x}_{\mathrm{i}}/x_{\mathrm{i}}$, we finally obtain the viability condition corresponding to the feasibility constraint $a^2 \leq \lambda \leq b^2$:

$$g\left(\frac{1}{a\tanh(aT_a))} - \frac{\sqrt{b^2 - a^2}}{ab\sinh(aT_a)}\right) \leq \dot{z}_{\mathrm{i}} - \frac{\dot{x}_{\mathrm{i}}}{x_{\mathrm{i}}}z_{\mathrm{i}} \leq g\left(\frac{\tanh(bT_b)}{b} + \frac{\sqrt{b^2 - a^2}}{ab\cosh(bT_b)}\right). \tag{27}$$

One could also invert these bounds to find the smallest and largest values for b and a, respectively, such that solutions exist.

References

[1] Stéphane Caron, Adrien Escande, Leonardo Lanari, and Bastien Mallein. Capturability-based analysis, optimization and control of 3D bipedal walking. In *IEEE-RAS International Conference on Robotics and Automation*, 2018.

[2] Stéphane Caron, Adrien Escande, Leonardo Lanari, and Bastien Mallein. Capturability-based analysis, optimization and control of 3d bipedal walking. Submitted, 2018.