Koopman DCM: Unstable Eigenfunctions as Data-driven Representations for Legged Balancing

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Model-based Representations for Balancing

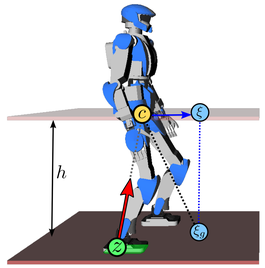
Balance = control the unstable mode of robot dynamics.

Legged dynamics: $M(q)\,\dot{v} + h(q,v) = S^\top \tau + J(q)^\top f$

Model reduction: CoM c and ZMP z

Linear inverted pendulum (LIP):

$$\ddot{c} = \omega^2 (c - z) \qquad \omega = \sqrt{\frac{g}{h}}$$



Eigendecomposition:

$$\underbrace{\dot{\xi} = \omega\,(\xi - z)}_{\text{divergent (DCM)}} \qquad \underbrace{\dot{c} = \omega\,(\xi - c)}_{\text{convergent}}$$

Low-dimensional: only the DCM ξ needs to be controlled.

Model-free Representations via Eigenfunctions

Legged dynamics are control-affine: $\dot{x} = f(x) + B(x)\,u$

The Koopman operator acts on observables $\varphi(x)$ as:

$$\mathscr{L}_f \varphi = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(x) = \nabla\varphi(x)\,f(x)$$

Koopman eigenfunctions $\mathscr{L}_f \varphi = \lambda\,\varphi$ satisfy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(x) = \lambda\,\varphi(x) + \nabla\varphi(x)\cdot B(x)\,u$$

Generalization of the DCM: search for large eigenvalues.

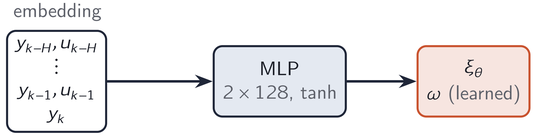
Low-dimensional: only learn the unstable part of the dynamics.

Koopman DCM

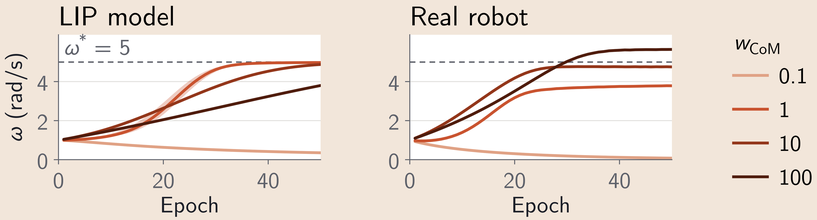
Search for unstable eigenpairs $(\omega,\ \xi_\theta)$ in robot transition data.

Neural network ξ_θ from embeddings rather than explicit states.

Delay embeddings of measurements y_k and actions u_k:



Koopman eigenfunction: $\dot{\xi}_\theta = \underbrace{\omega\,\xi_\theta}_{\text{eigenfunction}} - \underbrace{\omega\,u}_{\text{LIP-like input}}$



Training Loss

DCM dynamics loss: $\ell^{(0)}_{\text{DCM}} = \mathbb{E}\left\|\xi_k - \gamma\,\xi_{k+1} - (1-\gamma)\,u_k\right\|^2$

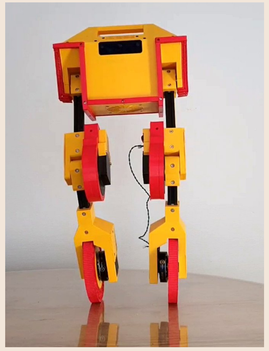
CoM prediction loss: $\ell^{(0)}_{\text{CoM}} = \mathbb{E}\left\|c_{k+1} - \gamma\,c_k - \frac{1-\gamma}{2}(\xi_k + \xi_{k+1})\right\|^2$

Weighted $\ell^{(0)} = \ell^{(0)}_{\text{DCM}} + w_{\text{CoM}}\,\ell^{(0)}_{\text{CoM}}$ to escape the trivial $\omega \to 0$.

Final loss: multi-step $\ell^{(\lambda)}$ with λ-return weights $(1-\lambda)\,\lambda^{n-1}$

Wheeled biped, 6 actuators

Measurements y_k	
q_k	joint angles
${}^A c_k$	anchor frame CoM
c_k	lateral CoM
Action u_k	
z_k	lateral ZMP

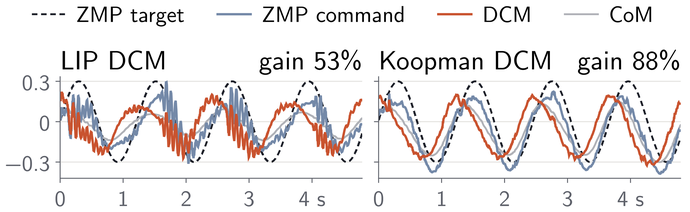


Results on Real Robot

no sim-to-real gap

Koopman DCM trained directly on 55 min of real-robot data:

	LIP	Learned LIP	Koopman DCM
ω (rad/s)	≈ 5.0	6.29	3.85



Model Predictive Control

DCMs give both dynamics and *viability*.

Viability constraint: $|\xi_k| \le \xi_{\max}$.

Optimizing future actions = convex QP.

Code & Data

github.com/stephane-caron/koopman_dcm

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